On the upper rate functions of some time inhomogeneous diffusion processes

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Abstract

In the present paper, we study an upper escape rate of some time inhomogeneous diffusion process associated with a family of regular and local Dirichlet forms. In particular, by making full use of Gaussian type's heat kernel estimates, we establish integral tests for an upper rate function of the time inhomogeneous diffusion process with a coefficient that is not necessarily bounded concerning space and time.

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1 Introduction

Let $\mathbb{B} = (B_t, \mathbb{P}_0)$ be the d-dimensional Brownian motion starting from the origin and $\varphi(t)$ be a positive increasing function such that $\lim_{t\to\infty} \varphi(t) = \infty$. If $\varphi(t)$ satisfies

$$\lim_{s\to\infty} \mathbb{P}_0\left(|B_t| > \varphi(t) \text{ for some } t \in [s,\infty)\right) = 0,$$

then $\varphi(t)$ is called an *upper rate function* of $\mathbb{B}$, which describes the forefront of the Brownian particles. It is well known as *Kolmogorov's test* that $\varphi(t)$ is an upper rate function of $\mathbb{M}$ if and only if

$$\int_1^\infty \left(\frac{\varphi(t)}{\sqrt{t}}\right)^d \exp\left(-\frac{\varphi(t)^2}{2t}\right)\frac{\mathrm{d}t}{t} < \infty \tag{1}$$

(see [9, §4.12]). In particular, the celebrated Khinchine's law of the iterated logarithm can be obtained by applying (1) to the function $\varphi(t) = \sqrt{(2+\varepsilon)t\log\log t}$ with $\varepsilon > 0$. Of course, there is a natural counterpart to an upper rate function, a lower rate function of $\mathbb{M}$, which is out of interest in the present paper.

There is extensive literature containing the problem that characterizes rate functions of various symmetric Markov processes. For instance, Grigor'yan [3], Grigor'yan and Kelbert [5], Grigor'yan and Hsu [4] and Hsu and Qin [6] obtained upper rate functions of the Brownian motion on a Riemannian manifold. Ouyang [13] characterized upper rate functions for symmetric diffusion processes. These results are further extended to the case of symmetric Markov processes associated with non-local Dirichlet forms (see [7, 8, 16, 17]).

The results cited so far were considered only for time homogeneous Markov processes. We are interested in studying the problem of characterizing the upper rate functions for time inhomogeneous cases. The aim of this paper is to investigate integral tests for an upper rate function of some time inhomogeneous diffusion process associated with a family of regular and local Dirichlet forms.

Let $(Z_t, \mathbb{P}_{(s_0,x_0)})$ with $Z_t = (t, B_t)$ be the space-time Brownian motion and Γ_s be a space-time domain given by $\Gamma_s = \{(\tau, x) : |x| = \varphi(\tau), \tau \geq s\}$. Denote by σ_{Γ_s} the first hitting time of Γ_s relative to Z_t. Then

$$\lim_{s\to\infty} \mathbb{P}_0\left(|B_t| > \varphi(t) \text{ for some } t \in [s,\infty)\right) = \lim_{s\to\infty} \mathbb{P}_{(0,0)}(\sigma_{\Gamma_s} < \infty).$$

In this way, the upper rate function is related to the hitting probability of the space-time process. Since the hitting probability is a potential of the equilibrium measure, it can be estimated by using the heat kernel and the capacity.

More generally, let $\mathbb{M} = (X_t, \mathbb{P}_{(s_0,x_0)})$ be a time inhomogeneous diffusion process on $\mathbb{R}^d$ with generator

$$\mathcal{L}^{(t)}u(t,x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u(t,x)}{\partial x_j}\right)$$

and $(Z_t, \mathbb{P}_{(s_0,x_0)})$ with $Z_t = (t, X_t)$ be its associated space-time process. Let us introduce a family of strongly local regular Dirichlet forms $\{(E^{(\tau)}, H^1(\mathbb{R}^d)), \tau \geq 0\}$ given by

$$E^{(\tau)}(u,v) := \sum_{i,j=1}^{d} a_{ij}(\tau,x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\mathrm{d}x, \quad u,v \in H^1(\mathbb{R}^d) \tag{2}$$

with a symmetric family of locally bounded measurable functions $(a_{ij}(\tau,x))_{i,j=1}^d$ on $[0,\infty)\times\mathbb{R}^d$. Here $H^1(\mathbb{R}^d)$ denotes the Sobolev space of order 1. Then the generator of Z_t is $\mathcal{L}u(\tau,x) = \partial u/\partial\tau + \mathcal{L}^{(\tau)}u(\tau,x)$ and its associated Dirichlet form on $L^2([0,\infty)\times\mathbb{R}^d;\nu)$ with $\mathrm{d}\nu(\tau,x) = \mathrm{d}\tau\mathrm{d}x$ is given by

$$\mathcal{E}(u,v) = -(\mathcal{L}u,v)_\nu = -\left(\frac{\partial u}{\partial \tau}, v\right)_\nu + \int_0^\infty E^{(\tau)}(u(\tau,\cdot), v(\tau,x))\mathrm{d}\tau \tag{3}$$

for any smooth functions u and v on $(0,\infty)\times\mathbb{R}^d$ with compact support. For any $\tau \geq 0$ and a relatively compact open set $D \subset \mathbb{R}^d$, we assume that there exist positive constants $\lambda(\tau,D)$ and $\Lambda(\tau,D)$ such that $\lambda_0 \leq \lambda(\tau,D)$ for a constant $\lambda_0 > 0$, and

$$\lambda(\tau,D)\sum_{i=1}^{d}\xi_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(\tau,x)\xi_i\xi_j \leq \Lambda(\tau,D)\sum_{i=1}^{d}\xi_i^2 \tag{4}$$

for all $\xi = (\xi_i, \dots, \xi_d) \in \mathbb{R}^d$ and $x \in D$. We say that a positive increasing function $\varphi(t)$ satisfying $\lim_{t\to\infty} \varphi(t) = \infty$ is an *upper rate function* of $\mathbb{M}$ if

$$\lim_{s\to\infty} \mathbb{P}_{(s_0,x_0)} \left(|X_t - x_0| > \varphi(t) \text{ for some } t \in [s, \infty)\right) = 0.$$

Note that we are not assuming the uniform boundedness of $\Lambda(\tau, D)$. As noted above, the upper rate function is related to the hitting probability of the associated space-time diffusion process. To evaluate the hitting probability, we use a new estimate of the capacity of the space-time process as well as the Gaussian type estimates. Under some additional assumptions on $\lambda(\tau, D)$ and $\Lambda(\tau, D)$, we shall give two-sided Gaussian type estimates of the time inhomogeneous transition densities (heat kernels) of $\mathbb{M}$ and its part process $\mathbb{M}^D$ on D. For this, we need to clarify the dependence of the space-time coefficients appearing in the Gaussian estimates. We shall prove it by modifying the results due to Lierl [10] and Sturm [18] (Proposition 3.1 and Proposition 3.2). The short summary of their proofs will be given as an appendix in Section 5.

In Section 3, for this specific time inhomogeneous diffusion processes, by making full use of these heat kernel estimates as well as a criterion for upper rate function given in Section 4, we give an integral test for an upper rate function of $\mathbb{M}$ (see Theorem 3.4 and Theorem 3.9). Some examples of the upper rate functions are also given. In the conditions and the results given in Section 3, instead of the original space-time function $\Lambda(t, D)$, we use the function $\Lambda(t, D(t))$ of t with a domain $D(t)$ of the form $D(t) = \{x : |x| < f(t)\}$ for a function $f(t)$. For the correspondence between the space-time functions $\Lambda(t, D)$ and the function $\Lambda(t, D(t))$ of t, see Corollary 3.5 and Example 3.7. In this section, we also give a condition on $\varphi(t)$ satisfying

$$\mathbb{P}_{(s_0,x_0)} \left(|X_t - x_0| > \varphi(t) \text{ for some } t \in [s, \infty)\right) = 1, \quad \text{for some } s > s_0.$$

In Section 4, we consider more general case that (E, d) is a locally compact separable metric space, m an everywhere dense positive Radon measure on E and $\{(E^{(\tau)}, F), \tau \geq 0\}$ is a family of strongly local regular Dirichlet forms on $L^2(E; m)$ defined by

$$E^{(\tau)}(u, v) = \int_E \mu^{(\tau)}_{\langle u,v\rangle}(\mathrm{d}x), \quad u, v \in F \tag{5}$$

with the energy measure $\mu^{(\tau)}_{\langle u,v\rangle}$ of u and v satisfying, for a relatively compact open subset $D \subset E$, there exist positive space-time coefficients $\lambda(\tau, D)$ and $\Lambda(\tau, D)$ such that

$$\lambda(\tau, D)\mu^{(0)}_{\langle u,u\rangle}(\mathrm{d}x) \leq \mu^{(\tau)}_{\langle u,u\rangle}(\mathrm{d}x) \leq \Lambda(\tau, D)\mu^{(0)}_{\langle u,u\rangle}(\mathrm{d}x) \tag{6}$$

for all $u \in F$ with $\mathrm{supp}[u] \subset D$. In this section, we give a general sufficient condition for $\varphi(t)$ being the upper rate function of $\mathbb{M}$ (Theorem 4.2). Integral representation of a hitting probability and the capacity estimate of the time-dependent Dirichlet form relative to $\{(E^{(\tau)}, F), \tau \geq 0\}$ play a crucial role (Lemma 2.1 and Lemma 4.1).

In the followings, we use the notations $K_1, K_2, \dots$ repeatedly to represent constants that are independent of time and space variables. But, even if the same notation K_i is used in the different places, they are not necessarily the same.

2 Preliminary results

In this section, we present necessary results on time dependent Dirichlet forms used in Section 4. Although we are giving them under general settings, for the concrete case considered in Section 3, it is enough to consider that $E = \mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$, $E^{(\tau)}$ and $\mathcal{E}$ are those given in (2) and (3), respectively.

Let (E, d) be a locally compact separable metric space and m be an everywhere dense positive Radon measure on E. Let $\{(E^{(\tau)}, F), \tau \geq 0\}$ be a family of regular Dirichlet forms on $H := L^2(E; m)$ such that

- For any $\varphi, \psi \in F$, $E^{(\tau)}(\varphi, \psi)$ is measurable with respect to τ.
- For any relatively compact open set $D \subset E$ and $\tau \geq 0$, there exist positive constants $\lambda(\tau, D) \leq \Lambda(\tau, D)$ satisfying $0 < \inf\{\lambda(\tau, D) : \tau \leq t\} \leq \sup\{\Lambda(\tau, D) : \tau \leq t\} < \infty$ and
$$\lambda(\tau, D)E^{(0)}(\varphi, \varphi) \leq E^{(\tau)}(\varphi, \varphi) \leq \Lambda(\tau, D)E^{(0)}(\varphi, \varphi), \tag{7}$$
for all $\varphi \in F$ with support in D.

We introduce the function spaces $\mathcal{H}$, $\mathcal{F}$ and $\mathcal{F}'$ given by $\mathcal{H} := L^2([0, \infty); H)$, $\mathcal{F} := L^2([0, \infty); F)$ and $\mathcal{F}' := L^2([0, \infty); F')$ respectively, that is

$$\mathcal{H} = \left\{ u : u(\tau, \cdot) \in H \text{ for all } \tau \geq 0,\ \|u\|_{\mathcal{H}}^2 := \int_0^\infty \|u(\tau, \cdot)\|_H^2 \mathrm{d}\tau < \infty \right\},$$

and $\mathcal{F}$, $\mathcal{F}'$ are defined similarly by taking F and F' instead of H respectively, where we consider $F \subset H \subset F'$ by identifying H with its dual space H'. For $u \in \mathcal{F}'$, let $\frac{\partial u}{\partial \tau}(\tau, \cdot)$ be the distribution sense derivative of $u(\tau, \cdot) \in F'$. Using this, we also define $\mathcal{W}$ by $\mathcal{W} = \left\{ u \in \mathcal{F} : \frac{\partial u}{\partial \tau} \in \mathcal{F}' \right\}$.

Define the bilinear form $\mathcal{E}$ by

$$\mathcal{E}(u, v) = \begin{cases} -\left(\dfrac{\partial u}{\partial \tau}, v \right) + \mathcal{A}(u, v), & u \in \mathcal{W}, v \in \mathcal{F} \\ \left(\dfrac{\partial v}{\partial \tau}, u \right) + \mathcal{A}(u, v), & u \in \mathcal{F}, v \in \mathcal{W}, \end{cases}$$

where $(\frac{\partial u}{\partial \tau}, v)$ is the coupling of $\frac{\partial u}{\partial \tau} \in \mathcal{F}'$ and $v \in \mathcal{F}$, that is, $(\frac{\partial u}{\partial \tau}, v) = \int_0^\infty {}_{F'}(\frac{\partial u}{\partial \tau}, v(\tau, \cdot))_F \, \mathrm{d}\tau$, and

$$\mathcal{A}(u, v) = \int_0^\infty E^{(\tau)}(u(\tau, \cdot), v(\tau, \cdot))\mathrm{d}\tau.$$

We call $(\mathcal{E}, \mathcal{F})$ the *time dependent Dirichlet form* on $\mathcal{H}$ corresponding to a family of Dirichlet forms $\{(E^{(\tau)}, F), \tau \geq 0\}$. It is known that there exists the space-time Hunt processs $\mathbb{M} = (Z_t, \mathbb{P}_z, z \in Z)$ with $Z_t = (t, X_t)$ corresponding to $(\mathcal{E}, \mathcal{F})$ (cf. [12, Theorem 6.3.10]). Here $Z = [0, \infty) \times E$. Let $p(s, x; t, \mathrm{d}y)$ be the time inhomogeneous transition function of $\mathbb{M}$ given by

$$p(s, x; t, B) = \mathbb{P}_{(s,x)}(X_t \in B), \quad 0 \leq s < t,\ x \in E,\ B \in \mathcal{B}(E).$$

Let $\widehat{\mathbb{M}} = (\widehat{Z}_t, \widehat{\mathbb{P}}_z, z \in Z)$ with $\widehat{Z}_t = (T-t, \widehat{X}_{T-t})$ be the dual process of $\mathbb{M}$. Denote by $\widehat{p}(t, y; s, \mathrm{d}x)$ the dual transition function $\widehat{\mathbb{M}}$. Define

$$P_{s,t}\psi(x) = \int_E p(s,x;t,\mathrm{d}y)\psi(y), \quad \widehat{P}_{t,s}\varphi(y) = \int_E \widehat{p}(t,y;s,\mathrm{d}x)\varphi(x). \tag{8}$$

Then, using the generator $\mathcal{L}^{(t)}$ corresponding to $E^{(t)}$, they satisfy

$$\frac{\partial}{\partial s} P_{s,t}\psi(x) = -\mathcal{L}^{(s)} P_{s,t}\psi(x), \quad \lim_{s\uparrow t} P_{s,t}\psi(x) = \psi(x) \tag{9}$$

$$\frac{\partial}{\partial t} \widehat{P}_{t,s}\varphi(y) = \mathcal{L}^{(t)} \widehat{P}_{t,s}\varphi(y), \quad \lim_{t\downarrow s} \widehat{P}_{t,s}\varphi(y) = \varphi(y). \tag{10}$$

Further, they are mutually in dual relative to m, that is

$$\int_E P_{s,t}\psi(x)\varphi(x)m(\mathrm{d}x) = \int_E \widehat{P}_{t,s}\varphi(y)\psi(y)m(\mathrm{d}y). \tag{11}$$

We assume that $\mathbb{M}$ and $\widehat{\mathbb{M}}$ satisfy the absolute continuity condition, that is, $\mathbb{M}$ and $\widehat{\mathbb{M}}$ admit the heat kernels $p(s,x;t,y)$ and $\widehat{p}(t,y;s,x)$ satisfying

$$p(s,x;t,\mathrm{d}y) = p(s,x;t,y)m(\mathrm{d}y) \quad \text{and} \quad \widehat{p}(t,y;s,\mathrm{d}x) = \widehat{p}(t,y;s,x)m(\mathrm{d}x)$$

in the strict sense, repectively (cf. [20, Theorem 2]). Thus we see that $\widehat{p}(t,y;s,x) = p(s,x;t,y)$ by (8) and (11).

For $\alpha \geq 0$, let $\{R_\alpha\}$ and $\{\widehat{R}_\alpha\}$ be the α-order resolvent and its dual resolvent of the space-time process $\mathbb{M}$ and $\widehat{\mathbb{M}}$ given by

$$R_\alpha f(s,x) = \int_0^\infty \int_E e^{-\alpha t} p(s,x;s+t,\mathrm{d}y) f(s+t,y)\mathrm{d}t$$

$$\widehat{R}_\alpha f(t,y) = \int_0^\infty \int_E e^{-\alpha s} \widehat{p}(t,y;t-s,\mathrm{d}x) f(t-s,x)\mathrm{d}s$$

for $f \in \mathcal{H}$, respectively. Note that $R_\alpha f$ is a quasi-continuous version of the potential $G_\alpha f$ satisfying

$$\mathcal{E}_\alpha(G_\alpha f, u) = (f, u), \quad u \in \mathcal{F}, \tag{12}$$

where $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)$. In particular, put $Rf = R_0 f$, $\widehat{R}f = \widehat{R}_0 f$ and $Gf = G_0 f$. For any $\alpha > 0$ and $f \in \mathcal{H}$, $G_\alpha f \in \mathcal{W}$ and satisfies

$$(f, G_\alpha f) = \mathcal{E}_\alpha(G_\alpha f, G_\alpha f) = \mathcal{A}_\alpha(G_\alpha f, G_\alpha f).$$

In particular, if $(f, Gf) < \infty$, this implies that $\alpha(G_\alpha f, G_\alpha f) \leq (f, G_\alpha f) \leq (f, Gf)$ and hence, $\alpha(G_\alpha f, G_\alpha f)$ and $\mathcal{A}(G_\alpha f, G_\alpha f)$ are bounded relative to α. This further implies that $Gf := \lim_{\alpha\to 0} G_\alpha f$ belongs to $\mathcal{F}_e$ and $\mathcal{A}(Gf, Gf) \equiv \lim_{\alpha\to 0} \mathcal{A}(G_\alpha f, G_\alpha f) = (f, Gf)$, where

$$\mathcal{F}_e = \left\{ u : u(\tau, \cdot) \in F_e \text{ for all } \tau \geq 0, \ \|u\|_{\mathcal{F}_e} := \mathcal{A}(u,u)^{1/2} < \infty \right\}$$

with the extended Dirichlet space F_e of F ([12, Chapter 1]). Since $\lim_{\alpha\to 0}\|\alpha G_\alpha f\|_{\mathcal{H}} = 0$, by letting $\alpha \to 0$ in (12), we obtain that $\mathcal{E}(Gf, u) = (f, u)$ for any $f \in \mathcal{H}$ with $(f, Gf) < \infty$ and $u \in \mathcal{F}$ such that $\frac{\partial u}{\partial \tau} \in \mathcal{F}_e$. This can be extended to $\mathcal{F}_e$, that is, for any f such that $(f, Gf) < \infty$,

$$\mathcal{E}(Gf, u) = (f, u) \qquad \text{for any } u \in \mathcal{W}_e,$$

where $\mathcal{W}_e = \{u \in \mathcal{F}_e : \frac{\partial u}{\partial \tau} \in \mathcal{F}_e\}$.

For $0 < s < t$ and a relatively compact open set $B \subset E$, let $\Gamma = (s, t) \times B$. It is known that for any $\varepsilon > 0$ there exists a unique function $u_\varepsilon \in \mathcal{W}$ satisfying

$$\left(\frac{\partial w}{\partial \tau}, u_\varepsilon\right) + \mathcal{A}(u_\varepsilon, w) = \frac{1}{\varepsilon}((u_\varepsilon - 1_\Gamma)^-, w), \quad w \in \mathcal{W} \tag{13}$$

(cf. [12, Lemma 6.2.3]). Note that u_ε increases and converges in $\mathcal{F}_e$ as $\varepsilon \downarrow 0$, to a function h_Γ satisfying

$$\left(\frac{\partial w}{\partial \tau}, h_\Gamma\right) + \mathcal{A}(h_\Gamma, w) \geq \mathcal{A}(h_\Gamma, h_\Gamma), \quad \text{for any } w \in \mathcal{W} \cap \mathcal{L}_{1_\Gamma}, \tag{14}$$

where $\mathcal{L}_{1_\Gamma} = \{u : u \geq 1_\Gamma\}$. This implies that h_Γ is the minimal excessive function which dominates 1_Γ a.e. Further, if $w \in \mathcal{W}$ satisfies $w = 0$ a.e. on Γ, then $\mathcal{E}(u_\varepsilon, w) = \frac{1}{\varepsilon}((u_\varepsilon - 1_\Gamma)^-, w) = 0$. Hence, h_Γ is a space-time harmonic function on $E \setminus \overline{\Gamma}$ in the sense that

$$\mathcal{E}(h_\Gamma, w) = 0 \qquad \text{for any } w \in \mathcal{W} \text{ with } \mathrm{supp}[w] \subset E \setminus \overline{\Gamma}. \tag{15}$$

According to the absolute continuity condition, $h_\Gamma(\sigma, x)$ has an everywhere defined excessive modification. We consider that h_Γ is a such modification. By using the space-time process $\mathbb{M}$, the excessive function h_Γ is characterized as

$$h_\Gamma(\tau, x) = \mathbb{P}_{(\tau,x)}(\sigma_\Gamma < \infty),$$

where $\sigma_\Gamma = \inf\{\sigma > 0 : Z_\sigma \in \Gamma\}$. Further, by the general theory of the equilibrium measure, there exists a unique positive Radon measure μ_Γ on $\overline{\Gamma}$ such that

$$h_\Gamma(\tau, x) = R\mu_\Gamma(\tau, x) = \iint_{\overline{\Gamma}} p(\tau, x; \sigma, y)\mu_\Gamma(\mathrm{d}\sigma \mathrm{d}y). \tag{16}$$

The capacity $\mathrm{Cap}(\Gamma)$ of Γ relative to the space-time process is given by $\mathrm{Cap}(\Gamma) = \mu_\Gamma(\overline{\Gamma})$. Then it satisfies

$$\mathrm{Cap}(\Gamma) = \iint_{\overline{\Gamma}} w(\sigma, y)\mu_\Gamma(\mathrm{d}\sigma \mathrm{d}y) = \mathcal{E}(R\mu_\Gamma, w) = \mathcal{E}(h_\Gamma, w) \geq \mathcal{A}(h_\Gamma, h_\Gamma) \tag{17}$$

for any $w \in \mathcal{W}_e$ such that $w = 1$ on Γ.

For $0 < p < q < \infty$ and a fixed $x_0 \in E$, let $B_p^q := \{x \in E : p < d(x) < q\}$, where $d(x) := d(x, x_0)$ is the distance between x and x_0. For $0 < s < t$, let $\Gamma := \Gamma_{s,p}^{t,q} = (s, t) \times B_p^q$. For the hitting probability $h_\Gamma \in \mathcal{F}$, there exists the equilibrium measure μ_Γ satisfying

$$\mathcal{E}(h_\Gamma, w) = \iint_Z w(\sigma, y)\mu_\Gamma(\mathrm{d}\sigma \mathrm{d}y) \quad \text{for all } w \in \mathcal{W}. \tag{18}$$

We say that $(\mathcal{E},\mathcal{F})$ has the strong local property if $(E^{(\tau)},F)$ possesses the strong local property for any $\tau \geq 0$. It is easy to see by (15) that μ_Γ is supported by

$$\partial\Gamma = \left(\{s\}\times\overline{B}_p^q\right)\cup\left([s,t]\times\partial B_p^q\right)\cup\left(\{t\}\times\overline{B}_p^q\right) \tag{19}$$

under the strong locality of $(\mathcal{E},\mathcal{F})$ (cf. [12, Lemma 6.3.9]). Set $\mu_\tau := 1_{\{\tau\}\times\overline{B_p^q}}\cdot\mu_\Gamma$ for $\tau = s,t$.

Lemma 2.1. *Suppose that $(\mathcal{E},\mathcal{F})$ has the strong local property. Then, for $\Gamma=(s,t)\times B_p^q$, it holds that $\mu_s = 0$ and $\mu_t(\mathrm{d}\sigma\mathrm{d}x)=\delta_{\{t\}}(\mathrm{d}\sigma)1_{\overline{B_p^q}}(x)m(\mathrm{d}x)$. In particular, for any $(\tau,x)\in Z$*

$$h_\Gamma(\tau,x) = \iint_{(s\vee\tau,t)\times\partial B_p^q} p(\tau,x;\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) + \int_{\overline{B_p^q}} p(\tau,x;t,y)m(\mathrm{d}y). \tag{20}$$

Proof. For simplicity, write $B = B_p^q$. For any $\delta > 0$, define

$$\xi_1(\sigma) = \begin{cases} \frac{1}{\delta}(\sigma - s + \delta)_+ & \sigma \leq s \\ \frac{1}{\delta}(s+\delta-\sigma)_+ & \sigma > s. \end{cases}$$

Let $w_1(\sigma,y)=\xi_1(\sigma)\phi(y)$ for a positive continuous function ϕ on E with support contained in B. Then, by (18),

$$\begin{aligned}\iint_Z w_1(\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) &= \int_{s-\delta}^{s+\delta}\int_E \xi_1(\sigma)\phi(y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) = \mathcal{E}(h_\Gamma,w_1)\\ &= \int_{s-\delta}^{s+\delta}\xi_1'(\sigma)(h_\Gamma(\sigma,\cdot),\phi)\,\mathrm{d}\sigma + \int_{s-\delta}^{s+\delta}\xi_1(\sigma)E^{(\sigma)}(h_\Gamma(\sigma,\cdot),\phi)\,\mathrm{d}\sigma\\ &:= (I)_\delta + (II)_\delta. \end{aligned} \tag{21}$$

Since

$$(II)_\delta \leq \int_{s-\delta}^{s+\delta}\left|E^{(\sigma)}(h_\Gamma(\sigma),\cdot),\phi)\right|\mathrm{d}\sigma \leq \mathcal{A}(h_\Gamma,h_\Gamma)^{1/2}\left(\int_{s-\delta}^{s+\delta}\Lambda(\sigma,B)E^{(0)}(\phi,\phi)\,\mathrm{d}\sigma\right)^{1/2} < \infty,$$

$(II)_\delta \to 0$ as $\delta\to 0$. On the other hand, since $h_\Gamma(\sigma,x)=\int_E p(\sigma,x;s,\mathrm{d}y)h_\Gamma(s,y)$ and $h_\Gamma(s,y)=1$ a.e. $y\in\overline{B}$ for $s-\delta<\sigma<s$

$$(1_{\overline{B}},\phi) \geq (h_\Gamma(\sigma,\cdot),\phi) \geq \left(\int_{\overline{B}} p(\sigma,\cdot;s,y)m(\mathrm{d}y),\phi\right) = \left(\widehat{P}_{s,\sigma}\phi, 1_{\overline{B}}\right)$$

and $\lim_{\sigma\uparrow s}\widehat{P}_{s,\sigma}\phi=\phi$, we see $\lim_{\sigma\uparrow s}(h_\Gamma(\sigma,\cdot),\phi)=(1_{\overline{B}},\phi)$. By using this fact, we have

$$\lim_{\delta\to 0}(I)_\delta = \lim_{\delta\to 0}\frac{1}{\delta}\left(\int_{s-\delta}^{s}(h_\Gamma(\sigma,\cdot),\phi)\,\mathrm{d}\sigma - \int_s^{s+\delta}(1_{\overline{B}},\phi)\,\mathrm{d}\sigma\right) = 0.$$

Therefore, by (21)

$$\iint_Z \phi(y)\mu_s(d\sigma dy) = \iint_Z \phi(y)1_{\{s\}\times\overline{B}}(\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y)$$
$$\le \lim_{\delta\to0}\int_{s-\delta}^{s+\delta}\int_E \xi_1(\sigma)\phi(y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) = \lim_{\delta\to0}(I)_\delta + \lim_{\delta\to0}(II)_\delta = 0,$$

hence we have $\mu_s = 0$. Similarly, replacing $\xi_1(\sigma)$ and $w_1(\sigma,y)$ by

$$\xi_2(\sigma) = \begin{cases} \frac{1}{\delta}(\sigma - t + \delta)_+ & \sigma \le t \\ \frac{1}{\delta}(t+\delta-\sigma)_+ & \sigma > t \end{cases}$$

and $w_2(\sigma,y) = \xi_2(\sigma)\phi(y)$ respectively, and noting that $h_\Gamma(\sigma,y) = 1$ for $(\sigma,y)\in(t-\delta,t)\times\overline{B}$ with $0<\delta<(t-s)/2$ and $h_\Gamma(\sigma,y)=0$ for $\sigma\ge t$, we have

$$\begin{aligned}\iint_Z \phi(y)\mu_t(\mathrm{d}\sigma\mathrm{d}y) &= \iint_Z \phi(y)1_{\{t\}\times\overline{B}}(\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y)\\
&= \lim_{\delta\to0}\int_{t-\delta}^{t+\delta}\int_E \xi_2(\sigma)\phi(y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) = \mathcal{E}(h_\Gamma, w_2)\\
&= \lim_{\delta\to0}\left(\frac{1}{\delta}\int_{t-\delta}^{t}\int_{\overline{B}}\phi(y)m(\mathrm{d}y)\mathrm{d}\sigma + \int_{t-\delta}^{t}\xi_2(\sigma)E^{(\sigma)}(h_\Gamma(\sigma,\cdot),\phi)\,\mathrm{d}\sigma\right)\\
&= \int_E \phi(y)\delta_{\{t\}}(\mathrm{d}\sigma)1_{\overline{B}}(y)m(\mathrm{d}y).\end{aligned}$$

The last assertion is a consequence of (16) with (19) and the first two assertions of the present lemma. Indeed,

$$\begin{aligned}h_\Gamma(\tau,x) &= \iint_{\overline{\Gamma}} p(\tau,x;\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) = \iint_{\partial\Gamma} p(\tau,x;\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y)\\
&= \iint_{\overline{\Gamma}} p(\tau,x;\sigma,y)\mu_s(\mathrm{d}\sigma\mathrm{d}y) + \iint_{(s\vee\tau,t)\times\partial B} p(\tau,x;\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y)\\
&\qquad + \iint_{\overline{\Gamma}} p(\tau,x;\sigma,y)\mu_t(d\sigma dy)\\
&= \iint_{(s\vee\tau,t)\times\overline{B}} p(\tau,x;\sigma,y)\mu_\Gamma(\mathrm{d}\sigma\mathrm{d}y) + \int_{\overline{B}} p(\tau,x;t,y)m(\mathrm{d}y).\end{aligned}$$

□

For $\ell>0$, set $B(\ell) := \{x\in E \mid d(x)<\ell\}$ and $\Gamma_\ell := \Gamma_s^t(\ell) = (s,t)\times\partial B(\ell)$. The following is an immediate result from the last assertion of Lemma 1 by applying Γ_ℓ to Γ.

Lemma 2.2. *Assume that $m(\partial B(\ell)) = 0$ for all $\ell > 0$. Then*

$$h_{\Gamma_\ell}(\tau,x) \le \sup_{(\sigma,y)\in\Gamma_\ell} p(\tau,x;\sigma,y)\,\mathrm{Cap}(\Gamma_\ell) \tag{22}$$

for any $(\tau,x)\in Z$.

3 Case of diffusion processes on $\mathbb{R}^d$

In this section, we consider a specific time inhomogeneous diffusion processes stated in the Introduction. Throughout this section, let $E = \mathbb{R}^d$ and m be the Lebesgue measure on $\mathbb{R}^d$, $m(\mathrm{d}x) = \mathrm{d}x$. Let $\{(E^{(\tau)}, H^1(\mathbb{R}^d)), \tau \geq 0\}$ be a family of strongly local regular Dirichlet forms given by (2), that is, whose energy measure $\mu^{(\tau)}_{\langle u,v\rangle}$ is given by

$$\mu^{(\tau)}_{\langle u,v\rangle}(\mathrm{d}x) = \sum_{i,j=1}^{d} a_{ij}(\tau, x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\mathrm{d}x, \quad u, v \in H^1(\mathbb{R}^d) \tag{23}$$

with a symmetric family of locally bounded measurable functions $(a_{ij}(\tau, x))_{i,j=1}^d$ on $[0, \infty) \times \mathbb{R}^d$. For any $\tau \geq 0$ and a relatively compact open set $D \subset \mathbb{R}^d$, we assume that there exist positive constants $\lambda(\tau, D)$ and $\Lambda(\tau, D)$ such that $\lambda_0 \leq \lambda(\tau, D)$ for a constant $\lambda_0 > 0$ and satisfying (4). For $0 < s < t$, let

$$\lambda(s, t, D) = \inf_{\tau\in[s,t]} \lambda(\tau, D), \quad \Lambda(s, t, D) = \sup_{\tau\in[s,t]} \Lambda(\tau, D).$$

Note that $\lambda(s, t, D)$ and $\Lambda(s, t, D)$ are decreasing and increasing relative to t for each fixed $s > 0$ and D, respectively. In the sequel, let $\mathbb{M} = (Z_t, \mathbb{P}_{z_0})$ be the associated space-time diffusion process of the time dependent Dirichlet form corresponding to $\{(E^{(\tau)}, H^1(\mathbb{R}^d)), \tau \geq 0\}$ and denote by $\mathbb{M}^D = (Z_t, \mathbb{P}^D_{z_0})$ the part process of $\mathbb{M} = (Z_t, \mathbb{P}_{z_0})$ on $[0, \infty) \times D$ with a starting point $z_0 = (s_0, x_0) \in [0, \infty) \times D$.

For fixed $R > 0$ and $b > 0$, let $p = (2R + b)/3$. Put $D = B(x_0, 3p) := B(3p)$. Define a new symmetric family of locally bounded functions $(\overline{a}_{ij}(\tau, x))_{i,j=1}^d$ by $\overline{a}_{ij}(\tau, x) = a_{ij}(\tau, \overline{x})$, where

$$\overline{x} = \left(\frac{3p}{|x|} \wedge 1\right) x, \quad x \in \mathbb{R}^d. \tag{24}$$

Clearly, $\overline{a}_{ij}(\tau, x) = a_{ij}(\tau, x)$ on D. Let denote by $\overline{E}^{(\tau)}(u, v)$ the Dirichlet form on $L^2(\mathbb{R}^d)$ given by (23) with the coefficient function $\overline{a}_{ij}(\tau, x)$ instead of $a_{ij}(\tau, x)$ and $\overline{\mathbb{M}} = (Z_t, \overline{\mathbb{P}}_{z_0})$ the space-time diffusion process on $[0, \infty) \times \mathbb{R}^d$ corresponding to the time dependent Dirichlet form associated with $\{(\overline{E}^{(\tau)}, H^1(\mathbb{R}^d)), \tau \geq 0\}$. Note that it makes no difference to replace $\mathbb{M}^D$ by the part process $\overline{\mathbb{M}}^D$ of $\overline{\mathbb{M}}$ on $[0, \infty) \times D$ because $\mathbb{M}^D$ is also the part process of $\overline{\mathbb{M}}$ on $[0, \infty) \times D$.

Let $\overline{p}(s, x; t, y)$ and $p^D(s, x; t, y)$ be the transition densities (heat kernels) of $\overline{\mathbb{M}}$ and $\mathbb{M}^D$, respectively. First we give two-sided estimates of the heat kernels $\overline{p}(s, x; t, y)$ and $p^D(s, x; t, y)$. The Gaussian estimates of heat kernels have been studied by many authors (see [1, 10, 11, 18, 19]). In this section, we shall use the results obtained by Lierl [10] and Sturm [18]. However, we want to emphasize in our case that $\Lambda(\tau, D)$ is not necessarily bounded relative to τ and D. Hence we need to modify their results to specify the dependence on τ and D of the constants appearing in the Gaussian estimates. By modifying their proofs, we can obtain the following theorems. We shall give short summary of the proofs as an appendix in the final section of this paper.

Let ν be an arbitrary constant satisfying $\nu > 2$ if $d = 1, 2$, and $\nu = d$ if $d \geq 3$. First, we present the upper estimate of the heat kernel $p^D(s, x; t, y)$.

Proposition 3.1. *For a fixed $R > 0$, let $D = B(5R/2)$. Assume that $2t\Lambda(0, t, D) \leq R^2$. Then there exists a constant $K_1 > 0$ such that*

$$\overline{p}(s, x; t, y) \leq \frac{K_1}{((t-s)\Lambda(s, t, D))^{d/2}} \left(\frac{\Lambda(s, t, D)}{\lambda(s, t, D)} \right)^{\nu/2} \times \left(1 \vee \frac{2|x - y|}{\sqrt{(t-s)\Lambda(s, t, D)}} \right)^d \exp\left(-\frac{|x-y|^2}{4(t-s)\Lambda(s, t, D)} \right) \tag{25}$$

for any $x, y \in B(R)$. As a consequence, this estimate also holds for $p^D(s, x; t, y)$.

In the sequel, let $\varphi(t)$ be a positive increasing function such that $\lim_{t\to\infty} \varphi(t) = \infty$. From now on, we take as $R = R(t) := 2\varphi(t)$ and $D = D(t) := B(5R(t)/2) = B(5\varphi(t))$. For notational simplicity, let

$$r(s, t) = \sqrt{2(t-s)\Lambda(s, t, D(t))}, \quad C_H(s, t) = \left(\frac{\Lambda(s, t, D(t))}{\lambda(s, t, D(t))} \right)^{2(\nu+1)}.$$

Define the function $\psi(t)$ on $[0, \infty)$ by

$$\psi(t) = \frac{\varphi(t)^2}{2t\Lambda(0, t, D(t))}. \tag{26}$$

We assume that $\Lambda(t, D(t))$ is increasing relative to t. Thus we always regard as $\Lambda(0, t, D(t)) = \Lambda(t, D(t))$. Further, we make the following assumption $\psi(t)$.

Assumption (A)

(A1) $\psi(t)$ is an increasing function satisfying $\lim_{t\to\infty} \psi(t)/t = 0$ and there exists a constants $\gamma > 1$ such that

$$\lim_{s\to\infty} \frac{\psi(\gamma s)}{\psi(s)} = \lim_{s\to\infty} \frac{\Lambda(\gamma s, D(\gamma s))}{\Lambda(s, D(s))} = \lim_{s\to\infty} \frac{\lambda(\gamma s, D(\gamma s))}{\lambda(s, D(s))} = 1.$$

(A2) There exists $\varepsilon_0 \in (0, 1)$ such that

$$\lim_{t\to\infty} \left\{ \psi(t) - \frac{\nu}{2(1-\varepsilon_0)} \log \Lambda(0, t, D(t)) \right\} = \infty.$$

Now, we are ready to present a lower estimate of the heat kernel $p^{D(t)}(s, x; t, y)$ in our notations.

Proposition 3.2. *Suppose that Assumption (A) holds for some $\gamma > 1$. Then, there exist constants $C_1 > 0$, $C_2 > 1$ and $T > 0$ satisfying*

$$p^{D(t)}(s,x;t,y) \geq \frac{C_1|x-y|^d}{\psi(t)^{2(\nu+1)+d/2}r(s,t)^{2d}C_H(s,t)^{\gamma-1}} \exp\left(-\frac{\gamma|x-y|^2}{r(s,t)^2}(1 \vee \log C_2 C_H(s,t))\right)$$

for all $x, y \in B(R(t))$ and $s, t \in (T, \infty)$ such that $s < t \leq \gamma s$. As a consequence, this estimate also holds for $\overline{p}(s,x;t,y)$.

To apply the criterion given by Theorem 4.2, we need to estimate $1 - C_\varphi(\sigma)$ for

$$\begin{aligned} C_\varphi(\sigma) = \sup\Big\{ \int_{D(\gamma\sigma)} & p^{D(\gamma\sigma)}(\tau, x; \tau', y)\Psi_p(|x|)\mathrm{d}x : \tau' \in (\sigma, \gamma\sigma), \\ & \tau \in ((1-\delta_0)\gamma^{-1}\sigma, (1-\delta_0)\sigma), y \in \partial B(p) \ \text{ for } p \in (\varphi(\gamma^{-1}\sigma), \varphi(\sigma))\Big\} \end{aligned}$$

with $\Psi_p(r) = \frac{2}{p}\left((r - p/2)_+ \wedge (3p/2 - r)_+\right)$.

Lemma 3.3. *Suppose that Assumption (A) holds for some $\gamma > 1$. Then, there exists $K > 0$ such that for any $\sigma \in (s, \gamma s)$ with large enough $s > 0$,*

$$1 - C_\varphi(\sigma) \geq \frac{K}{\psi(\sigma)^{2\nu+(5+d)/2}C_H((1-\delta_0)\gamma^{-1}\sigma, \gamma\sigma)^{\gamma-1}(\log C_2 C_H((1-\delta_0)\gamma^{-1}\sigma, \gamma\sigma))^{d+1/2}}$$

for $0 < \delta_0 < 1$.

Proof. For $s > 0$ and $\gamma > 1$ enjoying Assumption (A), let $y \in D := D(\gamma^2 s)$ be a point such that $|y| = \varphi(s)$. For simplicity, put $p = \varphi(s)$. Let $\mathbf{n} = -y/|y|$ be the inward normal vector of $\partial B(p)$ at y and $\mathbf{e}$ be a unit vector satisfying $1/2 < \mathbf{n}\cdot\mathbf{e} =: \kappa_0 < 1$. Define

$$B := \{x \in B(p) \cap B(y, p/2) : (x-y)\cdot\mathbf{n} > \kappa_0|x-y|\}.$$

Let Ψ_p be the function given in the definition of $C(s, t, \sigma_0, p)$. We consider that $\sigma_0 = \delta_0 s$ for some $\delta_0 \in (0,1)$ and $t = \gamma s$. Then

$$1 - \Psi_p(|x|) = \frac{2}{p}(p - |x|)_+ \wedge 1 \quad \text{for } x \in B.$$

Let $\sigma \in (s, \gamma s)$. For any $\tau \in ((1-\delta_0)\gamma^{-1}\sigma, (1-\delta_0)\sigma)$ and $\tau' \in (\sigma, \gamma\sigma)$, put

$$G(\tau, \tau') = \frac{r(\tau, \tau')^2}{I(\tau, \tau')}, \quad b_G = b_G(\tau, \tau') = \sqrt{G(\tau, \tau')},$$

where $I(\tau, \tau') = \gamma(1 \vee \log C_2 C_H(\tau, \tau'))$. Note that

$$\frac{\Lambda(\tau, \gamma^2 s, D(\gamma^2 s))}{\Lambda(0, s, D(s))} \longrightarrow 1, \quad \text{as } t \to \infty$$

under (A1) of Assumption (A) and $\delta_0 s < \tau' - \tau < \delta_1 s$ with $\delta_1 := (\delta_0 - 1 + \gamma)\gamma$. Then

$$\frac{b_G}{p} = \frac{\sqrt{2(\tau'-\tau)\Lambda(\tau,\tau',D(\tau'))}}{\varphi(s)\sqrt{I(\tau,\tau')}} \le \sqrt{\frac{2\delta_1 s\Lambda(\tau,\gamma^2 s, D(\gamma^2 s))}{s\psi(s)\Lambda(0,s,D(s))I(\tau,\tau')}}$$
$$\le \frac{\sqrt{2\delta_1}K_2(\delta)}{\sqrt{\psi(s)I(\tau,\tau')}} \longrightarrow 0 \quad \text{as } s \to \infty. \tag{27}$$

From this, we may regard as $b_G < p/2$ for large s. Let $z_0 := y + b_G\mathbf{e}$. We note that $b_{z_0} := |z_0| = \sqrt{p^2 + b_G^2 - 2pb_G\kappa_0} > p - b_G$. Then, $|x| \le b_{z_0}$ for any $x \in B \setminus B(y, b_G)$ and

$$1 - \Psi_p(|x|) = \frac{2}{p}(p - |x|)_+ \wedge 1 \ge \frac{2}{p}(p - b_{z_0}),$$

hence

$$\int_{D(\gamma\sigma)} p^{D(\gamma\sigma)}(\tau,x;\tau',y)\Psi_p(|x|)\mathrm{d}x \le \int_D p^D(\tau,x;\tau',y)\Psi_p(|x|)\mathrm{d}x$$
$$\le \int_D p^D(\tau,x;\tau',y)\mathrm{d}x - \int_B (1-\Psi_p(|x|))p^D(\tau,x;\tau',y)\mathrm{d}x$$
$$\le 1 - \int_{B\setminus B(y,b_G)} \frac{2}{p}(p-b_{z_0})p^D(\tau,x;\tau',y)\mathrm{d}x. \tag{28}$$

Let $S_1 = \{\theta : |\theta| = 1, \mathbf{n}\cdot\theta > \kappa_0\}$ and $|S_1|$ be its surface area. Then, we have by Proposition 3.2 with the change of variable $x - y = r\theta$ for $r > b_G$ and $\theta \in S_1$ that

$$\int_{B\setminus B(y,b_G)} \frac{2}{p}(p-b_{z_0})p^D(\tau,x;\tau',y)\mathrm{d}x$$
$$\ge \frac{2C_1}{p}(p-b_{z_0})\int_{B\setminus B(y,b_G)} \frac{|x-y|^d}{\psi(\tau')^{2(\nu+1)+d/2}r(\tau,\tau')^{2d}C_H(\tau,\tau')^{\gamma-1}} e^{-\frac{|x-y|^2}{r(\tau,\tau')^2}I(\tau,\tau')}\mathrm{d}x$$
$$\ge \frac{2C_1}{p}(p-b_{z_0})\int_{S_1} d\theta \left(\int_{b_G}^{p/2} \frac{r^{2d-1}}{\psi(\tau')^{2(\nu+1)+d/2}r(\tau,\tau')^{2d}C_H(\tau,\tau')^{\gamma-1}} e^{-\frac{r^2}{G(\tau,\tau')}}\mathrm{d}r\right)$$
$$= \frac{2C_1|S_1|(p-b_{z_0})}{p\psi(\tau')^{2(\nu+1)+d/2}C_H(\tau,\tau')^{\gamma-1}I(\tau,\tau')^d}\int_1^{p/2b_G} r^{2d-1}e^{-r^2}\mathrm{d}r. \tag{29}$$

The last integral of the righthand side of (29) is bounded from below by a positive constant because $p/2b_G \to \infty$ as $s \to \infty$ in view of (27). Further, since

$$\sqrt{1 + (b_G/p)^2 - 2\kappa_0(b_G/p)} < \frac{\sqrt{5}}{2}$$

and $(2\kappa_0 - (b_G/p))(b_G/p) > b_G/(2p)$ for large s, one has

$$\frac{p-b_{z_0}}{p} = 1 - \sqrt{1 + (b_G/p)^2 - 2\kappa_0(b_G/p)}$$
$$= \frac{2\kappa_0(b_G/p) - (b_G/p)^2}{1 + \sqrt{1 + (b_G/p)^2 - 2\kappa_0(b_G/p)}} \ge \frac{C_3 b_G}{p} \ge \frac{C_4}{\sqrt{\psi(s)I(\tau,\tau')}}.$$

Hence, we obtain by (28) and the definition of $C_\varphi(\cdot)$ that for any $\sigma \in (s, \gamma s)$ with large enough s

$$\begin{aligned}
1 - C_\varphi(\sigma) &= \inf\Big\{1 - \int_{D(\gamma\sigma)} p^{D(\gamma\sigma)}(\tau, x; \tau', y)\Psi_p(|x|)\mathrm{d}x : \tau' \in (\sigma, \gamma\sigma), \\
&\qquad \tau \in ((1-\delta_0)\gamma^{-1}\sigma, (1-\delta_0)\sigma), y \in \partial B(p) \ \text{ for } p \in (\varphi(\gamma^{-1}\sigma), \varphi(\sigma))\Big\} \\
&\geq \inf\Big\{\int_{B\setminus B(y,b_G)} \frac{2(p - b_{z_0})}{p} p^D(\tau, x; \tau', y)\mathrm{d}x : \tau' \in (\sigma, \gamma\sigma), \\
&\qquad \tau \in ((1-\delta_0)\gamma^{-1}\sigma, (1-\delta_0)\sigma), y \in \partial B(p) \ \text{ for } p \in (\varphi(\gamma^{-1}\sigma), \varphi(\sigma))\Big\} \\
&\geq \inf\Big\{\frac{C_5}{\psi(\tau')^{2(\nu+1)+d/2+1/2} C_H(\tau, \tau')^{\gamma-1} I(\tau, \tau')^{d+1/2}} : \tau' \in (\sigma, \gamma\sigma), \\
&\qquad \tau \in ((1-\delta_0)\gamma^{-1}\sigma, (1-\delta_0)\sigma), y \in \partial B(p) \ \text{ for } p \in (\varphi(\gamma^{-1}\sigma), \varphi(\sigma))\Big\} \\
&\geq \frac{C_6}{\psi(\sigma)^{2(\nu+1)+d/2+1/2} C_H((1-\delta_0)\gamma^{-1}\sigma, \gamma\sigma)^{\gamma-1} (\log C_2 C_H((1-\delta_0)\gamma^{-1}\sigma, \gamma\sigma))^{d+1/2}}.
\end{aligned}$$

The proof is complete. □

Theorem 3.4. *Suppose that Assumption (A) holds. Let* $\nu_d := 4(\nu+1) + 2d$ *and*

$$F_{d,\nu}(0,\sigma) := (\log C_2 C_H(0,\sigma) \vee 1)^{2d+1} \, \Lambda(0, \sigma, D(\sigma))^{\nu/2 + 4(\nu+1)(\gamma-1)}.$$

If $\psi(t)$ *satisfies for* $\varepsilon_0 \in (0,1)$ *in Assumption (A)*

$$\int_1^\infty \frac{\psi(\sigma)^{\nu_d} F_{d,\nu}(0,\sigma)}{\sigma} \exp\left(-\frac{(1-\varepsilon_0)}{2}\psi(\sigma)\right) \mathrm{d}\sigma < \infty, \tag{30}$$

then

$$\lim_{s\to\infty} \mathbb{P}_{(s_0,x_0)}\left(|X_\sigma - x_0| > \varphi(\sigma) \text{ for some } \sigma \in [s, \infty)\right) = 0.$$

In other words, $\varphi(t)$ *is an upper rate function of the time inhomogeneous diffusion process* X_t *associated with* (23).

Proof. Without loss of generality, let $(s_0, x_0) = (0, 0)$. We shall apply the general result obtained in Theorem 4.2. Let $\varepsilon_0 \in (0,1)$ be given in (A2) of Assumption (A). Take $r > 0$ so that $r/r(0,\sigma)$ being large. Then, by (25),

$$\begin{aligned}
\overline{\mathbb{P}}_{(0,0)}(X_\sigma \notin B(r/2)) &= \int_{B(r/2)^c} \overline{p}(0, 0; \sigma, y)\mathrm{d}y \\
&\leq \int_{B(r/2)^c} \frac{K_1 C_H(0,\sigma)^{\nu/4(\nu+1)}}{r(0,\sigma)^d} \left(\frac{2|y|}{r(0,\sigma)}\right)^d \exp\left(-\frac{|y|^2}{2r(0,\sigma)^2}\right) \mathrm{d}y \\
&\leq K_2 C_H(0,\sigma)^{\nu/4(\nu+1)} \int_{|z| > r/\sqrt{2}r(0,\sigma)} |z|^d e^{-\frac{|z|^2}{2}} \mathrm{d}z \\
&\leq K_3 C_H(0,\sigma)^{\nu/4(\nu+1)} \exp\left(-\frac{(1-\varepsilon_0)r^2}{4r(0,\sigma)^2}\right).
\end{aligned}$$

If we take $r = 2\varphi(\sigma)$, then by (A2) of Assumption (A), the last term of the above inequality is dominated by

$$\begin{aligned}
&K_3 C_H(0,\sigma)^{\nu/4(\nu+1)} \exp\left(-(1-\varepsilon_0)\psi(\sigma)\right) \\
&\quad \le K_3 \lambda_0^{-\nu/2} \exp\left(-\left((1-\varepsilon_0)\psi(\sigma) - \frac{\nu}{2}\log\Lambda(0,\sigma,D(\sigma))\right)\right) \\
&\quad \le K_4 \exp\left(-\left((1-\varepsilon_0)\psi(\sigma) - \frac{\nu}{2}\log\Lambda(0,\sigma,D(\sigma))\right)\right) \equiv \overline{H}(\sigma) \longrightarrow 0, \quad \text{as} \ \ \sigma \to \infty.
\end{aligned}$$

By applying this fact to [15, Lemma 3], we can confirm

$$\begin{aligned}
\mathbb{P}_{(0,0)}(|X_\tau| > \varphi(s), \exists \tau < s) &= \overline{\mathbb{P}}_{(0,0)}(|X_\tau| > \varphi(s), \exists \tau < s) \\
&\le \overline{\mathbb{P}}_{(0,0)}(\tau_{B(\varphi(s)/2)} \le s) \le \frac{\overline{H}(s)}{1-\overline{H}(s)} \longrightarrow 0, \quad \text{as} \ \ s \to \infty,
\end{aligned}$$

which implies (54).

Next, we will prove the integral condition (55). For any $\varepsilon > 0$, let us take s being large enough. Then we see from (25) with (A1) of Assumption (A) that for any $\sigma > s$,

$$\begin{aligned}
P_\varphi(0,0;\sigma) &= \sup\left\{p^{D(\gamma\sigma)}(0,0;\tau,y) : (\tau,y) \in (\gamma^{-1}\sigma, \gamma\sigma) \times B_{\varphi(\gamma^{-1}\sigma)}^{\varphi(\gamma\sigma)}\right\} \\
&\le \frac{K_5 C_H(0,\gamma\sigma)^{\nu/4(\nu+1)}}{r(0,\gamma^{-1}\sigma)^d}\left(\frac{\varphi(\gamma\sigma)}{r(0,\gamma^{-1}\sigma)}\right)^d \exp\left(-\frac{\varphi(\gamma^{-1}\sigma)^2}{2r(0,\gamma\sigma)^2}\right) \\
&\le \frac{K_6 C_H(0,\sigma)^{\nu/4(\nu+1)}}{r(0,\sigma)^d}\psi(\sigma)^{d/2} \exp\left(-\frac{(1-\varepsilon_0)}{2}\psi(\sigma)\right). \qquad (31)
\end{aligned}$$

Further, one can see for such $\sigma > s$ that

$$\Lambda(\gamma;\sigma)\mu^{(0)}_{\langle|\cdot|,|\cdot|\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right) \le K_7 \Lambda(\sigma, D(\sigma))\varphi(\sigma)^d, \qquad (32)$$

where $\Lambda(\gamma;\sigma)$ is the function defined in Theorem 4.2. Then, by using (31), (32), (A2) in Assumption (A) and Lemma 3.3,

$$\begin{aligned}
&\int_s^\infty \frac{P_\varphi(0,0;\sigma)\Lambda(\gamma;\sigma)}{\varphi(\gamma^{-1}\sigma)^2(1-C_\varphi(\sigma))^2}\,\mu^{(0)}_{\langle|\cdot|,|\cdot|\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right)\mathrm{d}\sigma \\
&\quad \le K_8 \int_s^\infty \frac{C_H(0,\sigma)^{\nu/4(\nu+1)}\psi(\sigma)^{d/2}e^{-\frac{(1-\varepsilon_0}{2}\psi(\sigma)}}{r(0,\sigma)^d\varphi(\sigma)^2(1-C_\varphi(\sigma))^2}\Lambda(\sigma,D(\sigma))\varphi(\sigma)^d d\sigma \\
&\quad \le K_9 \int_s^\infty \frac{\psi(\sigma)^{d-1}}{(1-C_\varphi(\sigma))^2}\frac{\Lambda(\sigma,D(\sigma))}{r(0,\sigma)^2}C_H(0,\sigma)^{\nu/4(\nu+1)}e^{-\frac{(1-\varepsilon_0)}{2}\psi(\sigma)}\mathrm{d}\sigma \\
&\quad \le K_{10} \int_s^\infty \frac{\psi(\sigma)^{4\nu+2d+4}}{\sigma}\Lambda(\sigma,D(\sigma))^{\nu/2}C_H(0,\sigma)^{2(\gamma-1)}(\log C_2 C_H(0,\sigma))^{2d+1}e^{-\frac{(1-\varepsilon_0)}{2}\psi(\sigma)}\mathrm{d}\sigma \\
&\quad \le K_{11} \int_s^\infty \frac{\psi(\sigma)^{4(\nu+1)+2d}F_{d,\nu}(0,\sigma)}{\sigma}e^{-\frac{(1-\varepsilon_0)}{2}\psi(\sigma)}\mathrm{d}\sigma.
\end{aligned}$$

Since $C_H(0,\sigma) \le K_{12}\Lambda(0,\sigma,D(\sigma))^{2(\nu+1)}$, the righthand side of the above inequality converges to 0 as $s \to \infty$ under the condition (30), hence (55) is satisfied. Now the assertion of the present theorem follows from Theorem 4.2 which will be stated in the next section. □

In Theorem 3.4, we did not use the space-time function $\Lambda(t, D)$ explicitly but used only the function $\Lambda(t, D(t))$ of t. To get an explicit correspondence between $\Lambda(t, D)$ and an upper rate function $\varphi(\sigma)$ is not easy in general. In this case, we shall give an explicit upper rate function which is larger than $\varphi(t)$. Write $\Lambda(\sigma, D) = \Lambda(\sigma, r)$ for $D = B(r)$. Let us assume that the associated upper rate function $\varphi(\sigma)$ satisfying $\Lambda(\sigma, D(\sigma)) = \Lambda(\sigma, 5\varphi(\sigma))$ and

$$\varphi(\sigma) = \left(2\sigma\Lambda(\sigma, 5\varphi(\sigma))\psi(\sigma)\right)^{1/2} \tag{33}$$

with a function $\psi(\sigma)$ satisfying Assumption (A) is determined. In particular, we consider that $\psi(\sigma) = \beta \log \Lambda(\sigma, \varphi(\sigma))$. For simplicity, we assume that $\lambda(\sigma, D(\sigma)) = \lambda_0$. For the upper rate function $\varphi(\sigma)$ satisfying (33), by Assumption (A), one can see $1 \le \Lambda(\sigma, \varphi(\sigma)) \le \sigma^\delta$ by taking large σ for any $\delta > 0$. Put $\Lambda_1(\sigma) = \Lambda(\sigma, 5)$ and $\Lambda_2(\sigma) = \Lambda(\sigma, 5\sigma^\delta)$ and assume that for $\gamma > 1$

$$\lim_{\sigma\to\infty} \frac{\Lambda_2(\gamma\sigma)}{\Lambda_2(\sigma)} = 1. \tag{34}$$

Since $\Lambda(\sigma, r)$ is non-decreasing relative to r, by (33),

$$\left(2\sigma\Lambda_1(\sigma)\psi(\sigma)\right)^{1/2} \le \varphi(\sigma) \le \left(2\sigma\Lambda_2(\sigma)\psi(\sigma)\right)^{1/2}. \tag{35}$$

For $\Lambda_2(\sigma)$, take a function $\psi_2(\sigma)$ satisfying (30). We may assume that $\psi_2(\sigma) \ge \psi(\sigma)$. Then $\varphi_2(\sigma) := \left(2\sigma\Lambda_2(\sigma)\psi_2(\sigma)\right)^{1/2} \ge \varphi(\sigma)$. Thus we have the following result.

Corollary 3.5. *Let $\varphi(\sigma)$ be an upper rate function corresponding to $\Lambda(\sigma, D(\sigma)) = \Lambda(\sigma, 5\varphi(\sigma))$ for $\Lambda(\sigma, r) = \Lambda(\sigma, B(r))$. Assume that $\Lambda_2(\sigma) = \Lambda(\sigma, 5\sigma^\delta)$ satisfies (34). Then an upper rate function $\varphi_2(\sigma)$ determined by $\Lambda_2(\sigma)$ satisfies $\varphi(\sigma) \le \varphi_2(\sigma)$. Further, if $\Lambda_1(\sigma)$ also satisfies (34) and the corresponding upper rate function $\varphi_1(\sigma)$ exists, then $\varphi_1(\sigma) \le \varphi(\sigma)$.*

Remark 3.6. The integral condition (30) also can be written as

$$\int_1^\infty \frac{\varphi(\sigma)^{2\nu_d}}{\sigma^{\nu_d}} \frac{F_{d,\nu}(0,\sigma)}{\Lambda(0,\sigma,D(\sigma))^{\nu_d}} \exp\left(-\frac{\varphi(\sigma)^2}{2\sigma}\frac{1-\varepsilon_0}{2\Lambda(0,\sigma,D(\sigma))}\right)\frac{1}{\sigma}\mathrm{d}\sigma < \infty \tag{36}$$

by the definition of $\psi(\sigma)$. Let us consider the case that $\lambda(0, \sigma, D(\sigma)) = \Lambda(0, \sigma, D(\sigma)) = 1$. Then, the diffusion process associated with (23) is the time homogeneous d-dimensional Brownian motion in view of (4). Further, since $F_{d,\nu}(0,\sigma) = K_2(\gamma) = 1$ and $\varepsilon_0 = 0$, the integral condition (36) differs from Kolmogorov's condition (1) at the power $2\nu_d$ instead of d. This arises from the Gaussian estimates we used here are not optimal in the Brownian motion case.

For a given function $\Lambda(\sigma)$ and a function $\psi(\sigma)$ satisfying the condition of Theorem 3.4, let us consider the case that an upper rate function $\varphi(\sigma)$ is given by $\varphi(\sigma) = (2\sigma\Lambda(\sigma)\psi(\sigma))^{1/2}$. In this case, although it is difficult to find an explicit space-time function $\Lambda(\sigma, r) = \Lambda(\sigma, B(r))$ such that $\Lambda(\sigma, 5\varphi(\sigma)) = \Lambda(\sigma)$, but an approximate form can be considered. We shall give these correspondence by an example.

Example 3.7. Assume that $\lambda(\sigma, D(\sigma)) = \lambda_0 > 0$ and $\Lambda(\sigma, D(\sigma)) = K(\log\sigma)^p$. For some $q > p$, put $\psi(\sigma) = \beta \log\log\sigma$ with $\beta = q\nu/2(1-\varepsilon_0)$. Then they satisfy Assumption (A). Noting that $F_{d,\nu}(0,\sigma) \le K_1(\log\log\sigma)^{2d+1}(\log\sigma)^{p(\nu/2+4(\nu+1)(\gamma-1))}$, we have by Theorem 3.4,

$$\varphi(\sigma) = (K\beta\sigma(\log\sigma)^p \log\log\sigma)^{1/2} \tag{37}$$

is an upper rate function if

$$\int_4^\infty \frac{(\log\log\sigma)^{\nu_d+2d+1}(\log\sigma)^{p(\nu/2+4(\nu+1)(\gamma-1))}}{\sigma} \exp\left(-\frac{q\nu}{4}\log\log\sigma\right) \mathrm{d}\sigma < \infty.$$

This holds if $q > (4/\nu))(1 + p(\nu/2 + 4(\nu+1)(\gamma-1)))$. If Assumption (A) holds for some $\gamma' > 1$, then it holds for all $1 < \gamma \le \gamma'$. Hence, by taking smaller γ, the above integral is finite if $q > (4/\nu)(1 + p\nu/2)$.

Next, let us consider the problem approximating a space-time function $\Lambda(\sigma, r)$ satisfying $\Lambda(\sigma, 5D(\sigma)) = K(\log\sigma)^p$. In particular we assume that $\Lambda(\sigma, D) = \xi(\sigma)f(r)$ for a function $f(r)$ and $D = B(r)$. Then $\Lambda(\sigma, D(\sigma)) = \xi(\sigma)f(5\varphi(\sigma))$. Hence, we need to find non-decreasing functions $\xi(\sigma)$ and $f(r)$ satisfying $\xi(\sigma)f(5\varphi(\sigma)) = K(\log\sigma)^p$ for $\varphi(\sigma)$ given by (37). There are many choices of such functions. As an example, take any positive non-decreasing function $\xi(\sigma)$ such that $(\log\sigma)^p/\xi(\sigma)$ is non-decreasing. Using such function $\xi(\sigma)$, it is enough to put $f(r) = K(\log\varphi^{-1}(r/5))^p/\xi(\varphi^{-1}(r/5))$. To obtain more concrete result, noting that $\sigma^{1/2} < \varphi(\sigma) < \sigma^{1+\delta}$ for any $\delta > 0$ and for large σ,

$$\xi(\sigma)f(5\sigma^{1/2}) \le \xi(\sigma)f(5\varphi(\sigma)) = \Lambda(\sigma, D(\sigma)) = K(\log\sigma)^p \le \xi(\sigma)f(5\sigma^{(1+\delta)/2}).$$

Therefore

$$K\left(\frac{2}{1+\delta}\right)^p \frac{(\log(r/5))^p}{\xi((r/5)^{2/(1+\delta)})} \le f(r) \le K2^p\frac{(\log(r/5))^p}{\xi((r/5)^2)}.$$

Since $\delta > 0$ is arbitrary, $f(r)$ can be considered approximately equal to $K2^p(\log(r/5))^p/\xi((r/5)^2)$.

Let $\varphi(t)$ be an upper rate function corresponding to $\Lambda(t, D(\sigma))$ given by Theorem 3.4. Take a positive non-decreasing function $\psi_1(t)$ satisfying

$$\psi_1(t) \le \psi(t) \quad \text{and} \quad \lim_{t\to\infty}\frac{\psi_1(\gamma t)}{\psi_1(t)} = 1 \tag{38}$$

for the same $\gamma > 1$ in Assumption (A) on $\psi(t)$. We shall be concerned with giving a condition on a function $\varphi_1(t) = (2\sigma\Lambda(\sigma, D(\sigma))\psi_1(\sigma))^{1/2}$ to satisfy

$$\mathbb{P}_{(s_0,x_0)}\left(|X_t - x_0| > \varphi_1(t) \text{ for some } t \in [s,\infty)\right) = 1 \tag{39}$$

for any $s > s_0$.

For $p > 0$, set $D = B(5p)$. Let $(E^{(\tau)}, F^D)$ be the part on D of the Dirichlet form $(E^{(\tau)}, F)$ and denote by $e_B^{(\tau),D}$ and $\mathrm{C}^{(\tau),D}(B)$ the (0-order) equilibrium potential and the (0-)capacity of $B \subset D$ relative to $(E^{(\tau)}, F^D)$, respectively. Note that $\mathrm{C}^{(\tau),D}(B)$ and $\mathrm{C}^{(0),D}(B)$ are related by

$$\mathrm{C}^{(\tau),D}(B) \geq \lambda(\tau, D)E^{(0)}\left(e_B^{(\tau),D}, e_B^{(\tau),D}\right) \geq \lambda(\tau, D)\mathrm{C}^{(0),D}(B).$$

Further, by comparing with the case of Brownian motion, it holds that for a constant $K_1 > 0$ and for any dimension $d \geq 1$

$$\mathrm{C}^{(0),D}(\partial B(p)) \geq K_1 p^{d-2}. \tag{40}$$

Lemma 3.8. *For $\Gamma := (s,t) \times B$ with $B \subset D$, we have*

$$\mathrm{Cap}^D(\Gamma) \geq \mathrm{C}^{(0),D}(B)(t-s)\lambda(s,t,D).$$

Proof. Since $h_\Gamma^D(\tau,\cdot) \in F^D$ and $h_\Gamma^D(\tau,\cdot) \geq 1$ a.e. on B for almost all $\tau \geq 0$, we have

$$\begin{aligned}\mathrm{Cap}^D(\Gamma) &\geq \mathcal{A}(h_\Gamma^D, h_\Gamma^D) \geq \int_s^t E^{(\tau)}(h_\Gamma(\tau,\cdot), (h_\Gamma(\tau,\cdot))\,\mathrm{d}\tau \\ &\geq \int_s^t E^{(\tau)}\left(e_B^{(\tau),D}, e_B^{(\tau),D}\right)\mathrm{d}\tau \geq \mathrm{C}^{(0),D}(B)(t-s)\lambda(s,t,D).\end{aligned}$$

□

Define a sequence $\{s_n\}$ by $s_n = \gamma^n$ for $\gamma > 1$. For given $0 < s < t$, let m and n be the smallest and largest integers such that $s \leq s_{2m-1}$ and $s_{2n+1} \leq t$, respectively. We introduce the following sets:

$$E_m^n = \bigcap_{j=m}^{n}\left\{X_{2m-1} < \varphi_1(s_{2m-1}), X_\sigma < \varphi_1(s_{2j+1}) \text{ for all } \sigma \in (s_{2j}, s_{2j+1}]\right\}$$

$$\widetilde{E}_m^n = \bigcap_{j=m}^{n-1}\left\{X_{2m} < \varphi_1(s_{2m}), X_\sigma < \varphi_1(s_{2j+2}) \text{ for all } \sigma \in (s_{2j+1}, s_{2j+2}]\right\}$$

In the sequel, we let $D(t) := B(5\varphi(t))$.

Theorem 3.9. *Suppose that Assumption (A) holds for an increasing functions $\psi(\sigma), \Lambda(\sigma, D(\sigma))$ and $\varphi(\sigma) = (2\sigma\Lambda(\sigma, D(\sigma))\psi(\sigma))^{1/2}$. Further, for a positive non-decreasing function $\psi_1(\sigma)$ satisfying (38) and $\gamma_0 := 4\gamma^3/(\gamma-1)$, if*

$$\int_{s_1}^{\infty} \frac{\mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}\Lambda(0,\sigma,D(\sigma))^{d/2+4(\nu+1)(\gamma-1)}} \exp\Big(-\gamma_0\psi_1(\sigma)(\log C_2 C_H(0,\sigma)\vee 1)\Big)\mathrm{d}\sigma = \infty, \tag{41}$$

then it holds that

$$\mathbb{P}_{(s_0,x_0)}\left(|X_t - x_0| > \varphi_1(t) \text{ for some } t \in [s,\infty)\right) = 1$$

for any $s > s_0$.

Proof. Without loss of generality, let $(s_0, x_0) = (0,0)$. Set $\Gamma^{(j)}_{\varphi_1(s_j)} = (s_{j-1}, s_j] \times \partial B(\varphi_1(s_j))$. For any $0 < s \le t$, let m, n be the numbers corresponding to s, t in the definition of E^n_m. Then

$$\begin{aligned}
&\mathbb{P}_{(0,0)}\left(X_\sigma < \varphi_1(\sigma),\ \forall \sigma \in (s,t)\right)\\
&\le \mathbb{P}_{(0,0)}\left(\bigcap_{j=m}^{n}\Big\{X_{2m-1} < \varphi_1(s_{2m-1}), X_\sigma < \varphi_1(s_{2j+1}), \forall\sigma\in(s_{2j}, s_{2j+1}]\Big\}\right)\\
&\le \mathbb{E}_{(0,0)}\left[\mathbb{P}^{D(s_{2n+1})}_{(s_{2n-1},X_{s_{2n-1}})}\left(X_\sigma < \varphi_1(s_{2n+1}),\ \forall\sigma\in(s_{2n}, s_{2n+1}]\right) : E^{n-1}_m\right]\\
&\le \sup_{x_{2n-1}\in\overline{B}(\varphi_1(s_{2n-1}))} \mathbb{P}^{D(s_{2n+1})}_{(s_{2n-1},x_{2n-1})}\left(\sigma_{\Gamma^{(n)}_{\varphi_1(s_{2n+1})}} = \infty\right)\cdot\mathbb{P}_{(0,0)}(E^{n-1}_m)\\
&= \left(1 - \inf_{x_{2n-1}\in\overline{B}(\varphi_1(s_{2n-1}))} \mathbb{P}^{D(s_{2n+1})}_{(s_{2n-1},x_{2n-1})}\left(\sigma_{\Gamma^{(n)}_{\varphi_1(s_{2n+1})}} < \infty\right)\right)\cdot\mathbb{P}_{(0,0)}(E^{n-1}_m)\\
&\le \prod_{j=m}^{n}\left(1 - \inf_{x_{2j-1}\in\overline{B}(\varphi_1(s_{2j-1}))} \mathbb{P}^{D(s_{2j+1})}_{(s_{2j-1},x_{2j-1})}\left(\sigma_{\Gamma^{(j)}_{\varphi_1(s_{2j+1})}} < \infty\right)\right)\cdot\mathbb{P}_{(0,0)}(E^m_m).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}_{(0,0)}(X_\sigma < \varphi_1(\sigma),\ \forall\sigma\in(s,t))\\
&\quad\le \prod_{j=m}^{n-1}\left(1 - \inf_{x_{2j}\in\overline{B}(\varphi_1(s_{2j}))} \mathbb{P}^{D(s_{2j+2})}_{(s_{2j},x_{2j})}\left(\sigma_{\Gamma^{(j)}_{\varphi_1(s_{2j+2})}} < \infty\right)\right)\cdot\mathbb{P}_{(0,0)}\left(\widetilde{E}^m_m\right).
\end{aligned}$$

Note that for a sequence $\{\beta_j\}$ with $0 < \beta_j < 1$, $\lim_{n\to\infty}\prod_{j=1}^n(1-\beta_j) \le \lim_{n\to\infty}\prod_{j=1}^n e^{-\beta_j} = e^{-\sum_{j=1}^\infty \beta_j} = 0$ whenever $\sum_{j=1}^\infty \beta_j = \infty$. Hence, if we could show that

$$\sum_{j=2m-1}^{\infty} \inf_{x_j\in\overline{B}(\varphi_1(\gamma^j))} \mathbb{P}^{D(\gamma^{j+2})}_{(\gamma^j,x_j)}\left(\sigma_{\Gamma^{(j)}_{\varphi_1(\gamma^{j+2})}} < \infty\right) = \infty, \tag{42}$$

then it follows that $\lim_{t\to\infty}\mathbb{P}_{(0,0)}(X_\sigma<\varphi_1(\sigma),\forall\sigma\in(s,t))=0$ for all $s>0$.

Now, let us prove (42). Let $\mu^{(1)}_{2k+1}$ be the equilibrium measure of $\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}$ relative to the part process $\mathbb{M}^{D(\gamma^{2k+1})}$, $k\in\mathbb{N}$. Then, similarly to Lemma 2.2, one has

$$\begin{aligned}
&\inf_{x_{2k-1}\in\overline{B}(\varphi_1(\gamma^{2k-1}))}\mathbb{P}^{D(\gamma^{2k+1})}_{(\gamma^{2k-1},x_{2k-1})}\left(\sigma_{\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}}<\infty\right)\\
&\quad\geq\inf_{x_{2k-1}\in\overline{B}(\varphi_1(\gamma^{2k-1}))}\iint_{\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}}p^{D(\gamma^{2k+1})}(\gamma^{2k-1},x_{2k-1};\sigma,y)\mu_{2k+1}(\mathrm{d}\sigma\mathrm{d}y)\\
&\quad\geq\left\{\inf_{x_{2k-1}\in\overline{B}(\varphi_1(\gamma^{2k-1})),\,(\sigma,y)\in\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}}p^{D(\gamma^{2k+1})}(\gamma^{2k-1},x_{2k-1};\sigma,y)\right\}\cdot\mathrm{Cap}^{D(\gamma^{2k+1})}\left(\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}\right).
\end{aligned}\tag{43}$$

Note that $|x|^de^{-|x|^2}$ is decreasing for $|x|>\sqrt{d/2}$. Further, since

$$2\varphi_1(\gamma^{2k+1})\geq|x_{2k-1}-y|\geq\varphi_1(\gamma^{2k+1})-\varphi_1(\gamma^{2k-1})\geq(\gamma-1)\varphi_1(\gamma^{2k-1})>\sqrt{d/2}$$

for any $x_{2k-1}\in B(\varphi_1(\gamma^{2k-1}))$ and $y\in\partial B(\varphi_1(\gamma^{2k+1}))$, we have from Proposition 3.2 that for $(\sigma,y)\in\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}$

$$\begin{aligned}
&p^{D(\gamma^{2k+1})}(\gamma^{2k-1},x_{2k-1};\sigma,y)\\
&\quad\geq\frac{C_1|x_{2k-1}-y|^d}{\psi(\gamma^{2k+1})^{2(\nu+1)+d/2}C_H(\gamma^{2k-1},\gamma^{2k+1})^{\gamma-1}r(\gamma^{2k-1},\gamma^{2k+1})^{2d}}\\
&\qquad\qquad\times\exp\left(-\frac{\gamma|x_{2k-1}-y|^2}{r(\gamma^{2k-1},\gamma^{2k})^2}\log C_2C_H(\gamma^{2k-1},\gamma^{2k+1})\right)\\
&\quad\geq\frac{C_3\varphi_1(\gamma^{2k-1})^d}{\psi(\gamma^{2k+1})^{2(\nu+1)+d/2}C_H(\gamma^{2k-1},\gamma^{2k+1})^{\gamma-1}r(\gamma^{2k-1},\gamma^{2k+1})^{2d}}\\
&\qquad\qquad\times\exp\left(-\frac{4\gamma\varphi_1(\gamma^{2k+1})^2}{r(\gamma^{2k-1},\gamma^{2k})^2}\log C_2C_H(\gamma^{2k-1},\gamma^{2k+1})\right)\\
&\quad\geq\frac{C_4\gamma^{(2k-1)d/2}\Lambda(0,\gamma^{2k-1},D(\gamma^{2k-1}))^{d/2}\psi_1(\gamma^{2k-1})^{d/2}}{\psi(\gamma^{2k+1})^{2(\nu+1)+d/2}(\gamma^{2k+1}-\gamma^{2k-1})^d\Lambda(\gamma^{2k-1},\gamma^{2k+1},D(\gamma^{2k+1}))^{d+4(\nu+1)(\gamma-1)}}\\
&\qquad\qquad\times\exp\left(-\frac{4\gamma^{2k+2}\Lambda(0,\gamma^{2k+1},D(\gamma^{2k+1}))\psi_1(\gamma^{2k+1})}{(\gamma^{2k}-\gamma^{2k-1})\Lambda(\gamma^{2k-1},\gamma^{2k},D(\gamma^{2k}))}\log C_2C_H(\gamma^{2k-1},\gamma^{2k+1})\right).
\end{aligned}$$

Then, by taking large k, we have

$$\begin{aligned}
&p^{D(\gamma^{2k+1})}(\gamma^{2k-1},x_{2k-1};\sigma,y)\\
&\quad\geq\frac{C_5\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}\Lambda(0,\sigma,D(\sigma))^{d/2+4(\nu+1)(\gamma-1)}}\exp\left(-\frac{4\gamma^3\psi_1(\sigma)}{\gamma-1}(\log C_2C_H(0,\sigma)\vee 1)\right)
\end{aligned}\tag{44}$$

for all $\sigma \in (\gamma^{2k}, \gamma^{2k+1}]$. On the other hand, since $C^{(0),D(\gamma^{2k+1})}(\partial B(\varphi_1(\gamma^{2k+1})))/C^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))$ is bounded from below and above by positive constants for $\sigma \in (\gamma^{2k}, \gamma^{2k+1}]$, one has by Lemma 3.8 that

$$\begin{aligned}\mathrm{Cap}^{D(\gamma^{2k+1})}\left(\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}\right) &\geq \lambda_0 \mathrm{C}^{(0),D(\gamma^{2k+1})}(\partial B(\varphi_1(\gamma^{2k+1})))\left(\gamma^{2k+1}-\gamma^{2k}\right)\\ &\geq C_6 \mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\left(\gamma^{2k+1}-\gamma^{2k}\right). \end{aligned} \tag{45}$$

Then, by combining (43), (44) and (45), we have for all $\sigma \in (\gamma^{2k}, \gamma^{2k+1}]$ with large k

$$\begin{aligned}&\inf_{x_{2k-1}\in \overline{B}(\varphi_1(\gamma^{2k-1}))} \mathbb{P}^{D(\gamma^{2k+1})}_{(\gamma^{2k-1},x_{2k-1})}\left(\sigma_{\Gamma^{(k)}_{\varphi_1(\gamma^{2k+1})}}<\infty\right)\\ &\quad\geq \frac{C_7 \mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))(\gamma^{2k+1}-\gamma^{2k})\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}C_H(0,\sigma)^{\gamma-1}\Lambda(0,\sigma,D(\sigma))^{d/2}}\exp\left(-\frac{4\gamma^3\psi_1(\sigma)}{\gamma-1}(\log C_2C_H(0,\sigma)\vee 1)\right)\\ &\quad\geq \int_{\gamma^{2k}}^{\gamma^{2k+1}} \frac{C_8 \mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}\Lambda(0,\sigma,D(\sigma))^{d/2+4(\nu+1)(\gamma-1)}}\\ &\qquad\qquad\times\exp\Big(-\gamma_0\psi_1(\sigma)(\log C_2C_H(0,\sigma)\vee 1)\Big)\mathrm{d}\sigma. \end{aligned} \tag{46}$$

Similarly, we also have

$$\begin{aligned}&\inf_{x_{2k}\in \overline{B}(\varphi_1(\gamma^{2k}))} \mathbb{P}^{D(\gamma^{2k+2})}_{(\gamma^{2k},x_{2k})}\left(\sigma_{\Gamma^{(k)}_{\varphi_1(\gamma^{2k+2})}}<\infty\right)\\ &\quad\geq \int_{\gamma^{2k+1}}^{\gamma^{2k+2}} \frac{C_8 \mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}\Lambda(0,\sigma,D(\sigma))^{d/2+4(\nu+1)(\gamma-1)}}\\ &\qquad\qquad\times\exp\Big(-\gamma_0\psi_1(\sigma)(\log C_2C_H(0,\sigma)\vee 1)\Big)\mathrm{d}\sigma. \end{aligned} \tag{47}$$

Therefore, we see by (46) and (47) that the lefthand side of (42) is larger than

$$\int_{\gamma^{2m}}^{\infty} \frac{C_9 \mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\psi_1(\sigma)^{d/2}}{\psi(\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}\Lambda(0,\sigma,D(\sigma))^{d/2+4(\nu+1)(\gamma-1)}}\exp\Big(-\gamma_0\psi_1(\sigma)(\log C_2C_H(0,\sigma)\vee 1)\Big)\mathrm{d}\sigma.$$

Hence we can obtain (42) under (41). □

Example 3.10. As Example 3.7, let $\Lambda(\sigma, D(\sigma)) = K(\log\sigma)^p$ $(p>0)$, $\psi(\sigma) = \beta\log\log\sigma$ for $\beta = q\nu/2(1-\varepsilon_0)$ with $q > (4/\nu)(1+p(\nu/2+4(\nu+1)(\gamma-1)))$ and $\varphi(\sigma) = (2\sigma\Lambda(\sigma, D(\sigma))\psi(\sigma))^{1/2}$. Then, by (40), $\psi_1(\sigma)$ satisfies (41) if

$$\begin{aligned}&\int_{s_1}^{\infty} \frac{\mathrm{C}^{(0),D(\sigma)}(\partial B(\varphi_1(\sigma)))\psi_1(\sigma)^{d/2}}{(\log\log\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}(\log\sigma)^{p(d/2+4(\nu+1)(\gamma-1))}}e^{-\gamma_0\psi_1(\sigma)\log C_2C_H(\sigma)}\mathrm{d}\sigma\\ &\quad\geq \int_{s_1}^{\infty} \frac{K_1\varphi_1(\sigma)^{d-2}\psi_1(\sigma)^{d/2}}{(\log\log\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}(\log\sigma)^{p(d/2+4(\nu+1)(\gamma-1))}}e^{-2\gamma_0(\nu+1)\psi_1(\sigma)\log\log\sigma}\mathrm{d}\sigma\\ &\quad\geq \int_{s_1}^{\infty} \frac{K_2\psi_1(\sigma)^{d-1}}{(\log\log\sigma)^{2(\nu+1)+d/2}\sigma^{d/2}(\log\sigma)^{p(d/2+4(\nu+1)(\gamma-1))}}e^{-2\gamma_0(\nu+1)\psi_1(\sigma)\log\log\sigma}\mathrm{d}\sigma = \infty. \end{aligned}$$

This holds if $\psi_1(\sigma) = \beta_1$ for $\beta_1 < (\gamma-1)/(8\gamma^3(\nu+1))\,(1-p(d/2+4(\nu+1)(\gamma-1)))$.

4 A criterion for the upper rate function

In this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a time dependent Dirichlet form on $\mathcal{H}$ corresponding to a family of strongly local regular Dirichlet forms $\{(E^{(\tau)}, F), \tau \geq 0\}$ on $L^2(E;m)$ given by

$$E^{(\tau)}(u,v) = \int_E \mu^{(\tau)}_{\langle u,v\rangle}(\mathrm{d}x), \quad u,v \in F, \tag{48}$$

where $\mu^{(\tau)}_{\langle u,v\rangle}$ is the energy measure of u and v satisfying that for a relatively compact open subset $D \subset E$ and for any $\tau \geq 0$ there exist positive constants $\lambda(\tau, D)$ and $\Lambda(\tau, D)$ such that

$$\lambda(\tau, D)\mu^{(0)}_{\langle u,u\rangle}(\mathrm{d}x) \leq \mu^{(\tau)}_{\langle u,u\rangle}(\mathrm{d}x) \leq \Lambda(\tau, D)\mu^{(0)}_{\langle u,u\rangle}(\mathrm{d}x) \tag{49}$$

for any $u \in F$ with support in D. In this case, the associated space-time Hunt process $\mathbb{M} = (Z_t, \mathbb{P}_z, z \in Z)$ of $(\mathcal{E}, \mathcal{F})$ is a diffusion process on $Z = [0,\infty) \times E$. For $0 < s < t$, let

$$\lambda(s,t,D) = \inf_{\tau\in[s,t]} \lambda(\tau, D), \quad \Lambda(s,t,D) = \sup_{\tau\in[s,t]} \Lambda(\tau, D). \tag{50}$$

We may and do assume that $\Lambda(s,t,D) \geq 1$ because, if otherwise, it suffices to consider $\Lambda(s,t,D)\vee 1$ instead of $\Lambda(s,t,D)$.

For a fixed starting point $z_0 = (s_0, x_0) \in [0,\infty) \times D$, let denote by $\mathbb{M}^D = (Z_t, \mathbb{P}^D_{z_0})$ the part process of $\mathbb{M} = (Z_t, \mathbb{P}_{z_0})$ on $[0,\infty) \times D$. Then, for a Borel set B such that $\overline{B} \subset D$,

$$\mathbb{P}_{z_0}(\sigma_{(s,t)\times B^c} < \infty) = \mathbb{P}^D_{z_0}(\sigma_{(s,t)\times B^c} < \infty)$$

for $s_0 \leq s < t$ and $x_0 \in B$. In view of this fact, we may consider it for the part process on D, as far as we consider $\mathbb{P}_{z_0}(\sigma_{(s,t)\times B^c} < \infty)$ for such sets.

For $p > 0$, take as $D = B(x_0, 3p) := B(3p)$. Denote by $\mathrm{Cap}^D(\cdot)$ the capacity relative to the part process $\mathbb{M}^D$. Other notions (such as the excessive function h_Γ, the heat kernel $p(s,x;t,y)$ and etc.) defined relative to $\mathbb{M}^D$ are also defined by using superfix D. First, we shall give an estmate of $\mathrm{Cap}^D(\Gamma_p)$ for $\Gamma_p = (s,t) \times \partial B(p)$. For $\delta > 0$ and a fixed $\sigma_0 > 0$, let

$$\Phi_{\delta,\sigma_0}(\tau) = \begin{cases} \frac{1}{\delta}(\tau - s + \sigma_0 + \delta)_+ & \tau < s - \sigma_0 \\ 1 & s - \sigma_0 \leq \tau \leq t \\ \frac{1}{\delta}(t + \delta - \tau)_+ & t \leq \tau, \end{cases}, \quad \Psi_p(r) = \begin{cases} \frac{2}{p}\left(r - \frac{p}{2}\right)_+ & r \leq p \\ \frac{2}{p}\left(\frac{3p}{2} - r\right)_+ & p < r. \end{cases}$$

Further, we set

$$C(s,t,\sigma_0,p) = \sup_{\sigma\in(s,t), y\in\partial B(p)} \int_D p^D(s-\sigma_0, x; \sigma, y)\Psi_p(d(x))m(\mathrm{d}x). \tag{51}$$

A function u on E is said to be locally in F ($u \in F_{\mathrm{loc}}$ in notation) if for any relatively compact open set $U \subset E$ there exists a function $u_U \in F$ such that $u = u_U$ m-a.e. on U ([2, Chapter 3]). Note that $\mu^{(\tau)}_{\langle u,u\rangle}$ is well-defined for any $u \in F_{\mathrm{loc}}$ and $\tau \geq 0$. In the rest of this section, we assume that the distance function $d(\cdot) := d(\cdot, x_0)$ belongs to F_{loc}.

Lemma 4.1. *It holds that*

$$\mathrm{Cap}^D(\Gamma_p) \le \frac{4}{p^2(1-C(s,t,\sigma_0,p))^2}\int_{s-\sigma_0}^t \mu^{(\tau)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3p/2}_{p/2}\right)\mathrm{d}\tau. \tag{52}$$

Proof. Put $w(\tau,x) = \Phi_{\delta,\sigma_0}(\tau)\Psi_p(d(x))$. Note that $h^D_{\Gamma_p}(\tau,x) = R^D\mu_{\Gamma_p}(\tau,x)$ is right-continuous relative to τ and vanishes for $\tau \ge t$. Then

$$\begin{aligned}\left(h^D_{\Gamma_p}, \frac{\partial w}{\partial \tau}\right) &= \frac{1}{\delta}\int_{s-\sigma_0-\delta}^{s-\sigma_0}\int_D h^D_{\Gamma_p}(\tau,x)\Psi_p(d(x))m(\mathrm{d}x)\mathrm{d}\tau \\ &= \frac{1}{\delta}\left(\int_{s-\sigma_0-\delta}^{s-\sigma_0}\iint_{\Gamma_p}\left\{\int_D p^D(\tau,x;\sigma,y)\Psi_p(d(x))m(\mathrm{d}x)\right\}\mu_{\Gamma_p}(\mathrm{d}\sigma\mathrm{d}y)\right)\mathrm{d}\tau.\end{aligned}$$

Hence we have

$$\begin{aligned}\lim_{\delta\to 0}\left(h^D_{\Gamma_p}, \frac{\partial w}{\partial \tau}\right) &= \iint_{\Gamma_p}\left\{\int_D p^D(s-\sigma_0,x;\sigma,y)\Psi_p(d(x))m(\mathrm{d}x)\right\}\mu_{\Gamma_p}(\mathrm{d}\sigma\mathrm{d}y) \\ &\le C(s,t,\sigma_0,p)\mu_{\Gamma_p}(\overline{\Gamma}_p) = C(s,t,\sigma_0,p)\mathrm{Cap}^D(\Gamma_p).\end{aligned}$$

On the other hand, since $\mathcal{A}(h^D_{\Gamma_p}, h^D_{\Gamma_p}) \le \mathrm{Cap}^D(\Gamma_p)$,

$$\begin{aligned}\mathcal{A}(h^D_{\Gamma_p}, w) &= \frac{2}{p}\int_{s-\sigma_0-\delta}^t \Phi_{\delta,\sigma_0}(\tau)\int_{B^{3p/2}_{p/2}} \mathrm{d}\mu^{(\tau)}_{\langle h^D_{\Gamma_p}(\tau,\cdot),d(\cdot)\rangle}(x)\,\mathrm{d}\tau \\ &\le \frac{2}{p}\mathcal{A}(h^D_{\Gamma_p}, h^D_{\Gamma_p})^{1/2}\left(\int_{s-\sigma_0-\delta}^t \Phi_{\delta,\sigma_0}(\tau)\int_{B^{3p/2}_{p/2}} \mathrm{d}\mu^{(\tau)}_{\langle d(\cdot),d(\cdot)\rangle}(x)\,\mathrm{d}\tau\right)^{1/2}\end{aligned}$$

which implies

$$\lim_{\delta\to 0}\mathcal{A}(h^D_{\Gamma_p}, w) \le \frac{2}{p}\mathrm{Cap}^D(\Gamma_p)^{1/2}\left(\int_{s-\sigma_0}^t \mu^{(\tau)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3p/2}_{p/2}\right)\mathrm{d}\tau\right)^{1/2}.$$

Therefore, we have by (17) that

$$\begin{aligned}\mathrm{Cap}^D(\Gamma_p) &= \lim_{\delta\to 0}\left\{\left(h^D_{\Gamma_p}, \frac{\partial w}{\partial \tau}\right) + \mathcal{A}(h^D_{\Gamma_p}, w)\right\} \\ &\le C(s,t,\sigma_0,p)\,\mathrm{Cap}^D(\Gamma_p) + \frac{2}{p}\mathrm{Cap}^D(\Gamma_p)^{1/2}\left(\int_{s-\sigma_0}^t \mu^{(\tau)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3p/2}_{p/2}\right)\mathrm{d}\tau\right)^{1/2}.\end{aligned}$$

This yields (52). □

Let $\varphi(\tau)$ be a positive increasing function such that $\lim_{\tau\to\infty}\varphi(\tau) = \infty$. Take $\gamma > 1$ and $0 < \delta_0 < 1$. For $\sigma \in (s, \gamma s)$, let $D(\sigma) = B(5\varphi(\sigma)) := B(x_0, 5\varphi(\sigma))$. Set

$$\begin{aligned}P_\varphi(s_0,x_0;\sigma) &= \sup\left\{p^{D(\gamma\sigma)}(s_0,x_0;\tau,y) : (\tau,y)\in(\gamma^{-1}\sigma,\gamma\sigma)\times B^{\varphi(\gamma\sigma)}_{\varphi(\gamma^{-1}\sigma)}\right\}, \\ C_\varphi(\sigma) &= \sup\left\{\int_{D(\gamma\sigma)} p^{D(\gamma\sigma)}(\tau,x;t,y)\Psi_p(d(x))m(\mathrm{d}x) : \tau\in((1-\delta_0)\gamma^{-1}\sigma,(1-\delta_0)\sigma),\right. \\ &\qquad\qquad \left. t\in(\sigma,\gamma\sigma), y\in\partial B(p)\ \text{ for } p\in(\varphi(\gamma^{-1}\sigma),\varphi(\gamma\sigma))\right\}.\end{aligned}$$

By considering as $t = \gamma s, \sigma_0 = \delta_0 s$ and $p = \varphi(s)$ in the definition of $C(s, t, \sigma_0, p)$, it holds that

$$C(s, \gamma s, \delta_0 s, \varphi(s)) \leq C_\varphi(\sigma) \tag{53}$$

for all $\sigma \in (s, \gamma s)$.

The main result of this section is as follows:

Theorem 4.2. *Suppose that $m(\partial B(\ell)) = 0$ for all $\ell > 0$. Assume*

$$\lim_{s\to\infty} \mathbb{P}_{(s_0,x_0)} (d(X_\tau) > \varphi(s) \text{ for some } \tau \in [s_0, s)) = 0. \tag{54}$$

Further assume that for some $\gamma > 1$,

$$\int_{s_0}^{\infty} \frac{P_\varphi(s_0, x_0; \sigma)\Lambda(\gamma; \sigma)}{\varphi(\gamma^{-1}\sigma)^2(1 - C_\varphi(\sigma))^2}\, \mu^{(0)}_{\langle d(\cdot),d(\cdot)\rangle} \left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right) \mathrm{d}\sigma < \infty, \tag{55}$$

where $\Lambda(\gamma; \sigma) := \Lambda((1 - \delta_0)\gamma^{-1}\sigma, \gamma\sigma, D(\gamma\sigma))$ for $0 < \delta_0 < 1$. Then

$$\lim_{s\to\infty} \mathbb{P}_{(s_0,x_0)} (d(X_\sigma) > \varphi(\sigma) \text{ for some } \sigma \in [s, \infty)) = 0.$$

In other words, $\varphi(t)$ is an upper rate function of the time inhomogeneous diffusion process X_t associated with $\{(E^{(\tau)}, F)\}_{\tau\geq 0}$.

Proof. For $j \geq 1$ and $\gamma > 1$, let $\{s_j\}$ be a sequence of real numbers given by $s_j = \gamma s_{j-1}$. Set $\Gamma^{(j)}_{\varphi(s_{j-1})} = (s_{j-1}, s_j) \times \partial B(\varphi(s_{j-1}))$. For $0 < s_0 < s < t$, let k and n be numbers such that $s_{k-1} \leq s < s_k$ and $s_{n-1} \leq t < s_n$. Note that $s_n = \gamma^{n-k+1}s_{k-1}$. Then

$$\begin{aligned}
&\{\omega \mid d(X_\sigma(\omega)) > \varphi(\sigma),\ \exists\sigma \in (s, t)\} \\
&\quad \subset \{\omega \mid d(X_\sigma(\omega)) > \varphi(\sigma),\ \exists\sigma \in (s_{k-1}, s_n]\} \\
&\quad \subset \{\omega \mid d(X_\tau(\omega)) \geq \varphi(s_{k-1}), \exists\tau \in [s_0, s_{k-1})\} \\
&\qquad \cup \{\omega \mid d(X_\tau(\omega)) < \varphi(s_{k-1}), \forall\tau \in [s_0, s_{k-1}], d(X_\sigma(\omega)) > \varphi(\sigma),\ \exists\sigma \in (s_{k-1}, s_n]\} \\
&\quad \subset \{\omega \mid d(X_\tau(\omega)) \geq \varphi(s_{k-1}), \exists\tau \in [s_0, s_{k-1})\} \cup \\
&\qquad \bigcup_{\ell=k}^{n} \{\omega \mid d(X_\tau(\omega)) < \varphi(s_{\ell-1}), \forall\tau \in [s_0, s_{\ell-1}], d(X_\sigma(\omega)) \geq \varphi(s_{\ell-1}),\ \exists\sigma \in (s_{\ell-1}, s_\ell]\} \\
&\quad \subset \{\omega \mid d(X_\tau(\omega)) \geq \varphi(s_{k-1}), \exists\tau \in [s_0, s_{k-1})\} \cup \left(\bigcup_{\ell=k}^{n} \left\{ \omega \mid \sigma_{\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}}(\omega) < \tau_{D(s_\ell)}(\omega) \right\} \right),
\end{aligned} \tag{56}$$

where $\tau_{D(s_\ell)} = \inf\{\tau > s_0 : X_\tau \notin D(s_\ell)\}$. Since $\varphi(\gamma^{-1}\sigma) \leq \varphi(s_{\ell-1}) \leq \varphi(\sigma)$ for $\sigma \in (s_{\ell-1}, s_\ell)$, $\ell = k, k+1, \cdots, n$, we have from Lemma 4.1, (49), (50) and (53) with $s = s_{\ell-1}$, $t = s_\ell$, $\sigma_0 = \delta_0 s_{\ell-1}$

and $p=\varphi(s_{\ell-1})$ that

$$\begin{aligned}\mathrm{Cap}^{D(s_\ell)}\left(\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}\right) &\le \frac{4\displaystyle\int_{(1-\delta_0)s_{\ell-1}}^{s_\ell}\mu^{(\tau)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3\varphi(s_{\ell-1})/2}_{\varphi(s_{\ell-1})/2}\right)\mathrm{d}\tau}{\varphi(s_{\ell-1})^2(1-C(s_{\ell-1},s_\ell,\delta_0 s_{\ell-1},\varphi(s_{\ell-1})))^2}\\ &\le \frac{4(\gamma-1+\delta_0)s_{\ell-1}\Lambda(\gamma;\sigma)}{\varphi(\gamma^{-1}\sigma)^2(1-C_\varphi(\sigma))^2}\mu^{(0)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right).\end{aligned}\tag{57}$$

Further, for $\sigma\in(s_{\ell-1},s_\ell),\ell=k,k+1,\cdots,n$

$$\sup_{(\tau,y)\in\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}} p^{D(s_\ell)}(s_0,x_0;\tau,y)\le P_\varphi(s_0,x_0;\sigma)\tag{58}$$

and $(\gamma-1+\delta_0)s_{\ell-1}=K(s_\ell-s_{\ell-1})$ for $K=1+\delta_0/(\gamma-1)$. Then, by virtue of Lemma 2.2 with (57) and (58), we have

$$\begin{aligned}\mathbb{P}_{(s_0,x_0)}\left(\sigma_{\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}}<\tau_{D(s_\ell)}\right) &= \mathbb{P}^{D(s_\ell)}_{(s_0,x_0)}\left(\sigma_{\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}}<\tau_{D(s_\ell)}\right) = h^{D(s_\ell)}_{\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}}(s_0,x_0)\\ &\le \left\{\sup_{(\tau,y)\in\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}} p^{D(s_\ell)}(s_0,x_0;\tau,y)\right\}\mathrm{Cap}^{D(s_\ell)}\left(\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}\right)\\ &\le \frac{K_1(s_\ell-s_{\ell-1})P_\varphi(s_0,x_0;\sigma)\Lambda(\gamma;\sigma)}{\varphi(\gamma^{-1}\sigma)^2(1-C_\varphi(\sigma))^2}\mu^{(0)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right)\\ &\le K_1\int_{s_{\ell-1}}^{s_\ell}\frac{P_\varphi(s_0,x_0;\sigma)\Lambda(\gamma;\sigma)}{\varphi(\gamma^{-1}\sigma)^2(1-C_\varphi(\sigma))^2}\mu^{(0)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right)\mathrm{d}\sigma\end{aligned}\tag{59}$$

for some positive constant K_1. Therefore, it follows from (56) and (59) that

$$\begin{aligned}&\mathbb{P}_{(s_0,x_0)}(d(X_\sigma)>\varphi(\sigma),\ \exists\sigma\in(s,t))\\ &\quad\le \mathbb{P}_{(s_0,x_0)}(d(X_\tau)\ge\varphi(s_{k-1}),\exists\tau\in[s_0,s_{k-1}))+\sum_{\ell=k}^{n}\mathbb{P}_{(s_0,x_0)}\left(\sigma_{\Gamma^{(\ell)}_{\varphi(s_{\ell-1})}}<\tau_{D(s_\ell)}\right)\\ &\quad\le \mathbb{P}_{(s_0,x_0)}(d(X_\tau)\ge\varphi(s_{k-1}),\exists\tau\in[s_0,s_{k-1}))\\ &\qquad\qquad + K_1\int_{s_{k-1}}^{\infty}\frac{P_\varphi(s_0,x_0;\sigma)\Lambda(\gamma;\sigma)}{\varphi(\gamma^{-1}\sigma)^2(1-C_\varphi(\sigma))^2}\mu^{(0)}_{\langle d(\cdot),d(\cdot)\rangle}\left(B^{3\varphi(\sigma)/2}_{\varphi(\gamma^{-1}\sigma)/2}\right)\mathrm{d}\sigma.\end{aligned}$$

The first and last terms of the righthand side of the above converge to 0 as $s\to\infty$ under the assumptions (54) and (55) respectively, because $s_{k-1}\to\infty$ whenever $s\to\infty$. □

5 Appendix - Gaussian estimates of heat kernels

In this appendix, we give short summary of the proofs of Proposition 3.1 and Proposition 3.2. Let $\{(E^{(\tau)},F),\tau\ge 0\}$ be a family of strongly local regular Dirichlet forms given by (48) on $L^2(\mathbb{R}^d;m)$

with the Lebesgue measure m on $\mathbb{R}^d$ satisfying the condition (49). Throughout this section, we use the notations given in Section 4.

For any $0 < s < t$ and a relatively compact open set $D \subset \mathbb{R}^d$, we say that $u \in L^2_{\mathrm{loc}}([0,\infty);F)$ is a weak subsolution (resp. solution, supersolution) of the heat equation for $E^{(\tau)}$ on $(s,t) \times D$ if for any $\tau \in (s,t)$ and $\psi \in F \cap C_0(\mathbb{R}^d)_+$

$$\int_{\mathbb{R}^d} \frac{\partial u(\tau,x)}{\partial \tau}\psi(x)\,\mathrm{d}x + E^{(\tau)}(u(\tau.\cdot),\psi) \le 0 \quad (\text{resp. } = 0,\ \ge 0).$$

We also say that $\widehat{u} \in L^2_{\mathrm{loc}}([0,\infty);F)$ is a weak co-subsolution (resp. co-solution, co-supersolution) of the heat equation for $E^{(\tau)}$ on $(s,t) \times D$ if for any $\tau \in (s,t)$ and $\psi \in F \cap C_0(\mathbb{R}^d)_+$

$$-\int_{\mathbb{R}^d} \frac{\partial \widehat{u}(\tau,x)}{\partial \tau}\psi(x)\,\mathrm{d}x + E^{(\tau)}(\widehat{u}(\tau.\cdot),\psi) \le 0 \quad (\text{resp. } = 0,\ \ge 0)$$

Here $C_0(\mathbb{R}^d)_+$ stands for the set of non-negative continuous function on $\mathbb{R}^d$ with compact support.

Put $B(x,r) = \{y : |x-y| < r\}$ and $B(r) = B(0,r)$. We mainly consider the subset Q of $(s,t) \times D$ defined by $Q := I \times B(x,r)$ for $x \in D$, where $I = (a - \eta r^2, a + \eta r^2)$ for $a \in (s,t)$, $\eta > 0$ and $r > 0$ such that $B(4r) \subset D$. More precisely, let $a = (s+t)/2$ and choose $\eta > 0$ satisfying $\eta r^2 \le (t-s)/2$ for $r > 0$ such that $B(4r) \subset D$. Further, for any $0 < \delta \le 1$, let

$$\begin{aligned} I^-_\delta &= (a - \delta\eta r^2, a), \quad & Q^-_\delta &= I^-_\delta \times B(x,\delta r) \\ I^+_\delta &= (a, a + \delta\eta r^2), \quad & Q^+_\delta &= I^+_\delta \times B(x,\delta r). \end{aligned}$$

Assume that u is a non-negative locally bounded weak subsolution of the heat equation for $E^{(\tau)}$ on $(s,t) \times D$. For $0 < \delta' < \delta \le 1$ and $p \ge 2$, let

$$\psi(y) = \frac{(\delta r - |x-y|)_+}{(\delta - \delta')r} \wedge 1, \quad \chi(\tau) = \left(1 - \frac{(a - \delta'\eta r^2 - \tau)_+}{(\delta-\delta')\eta r^2}\right)_+.$$

Then, for $s_0 = a - \delta\eta r^2$ and $t_0 \in I^-_{\delta'}$,

$$\int_{\mathbb{R}^d} u(t_0,y)^p \psi(y)^2 m(\mathrm{d}y) = \int_{s_0}^{t_0}\int_{B(x,\delta r)} \frac{\partial}{\partial \tau}\left(u(\tau,y)^p\psi(y)^2\chi(\tau)\right) m(\mathrm{d}y)\mathrm{d}\tau.$$

Similarly to the proof of [10, Lemma 1.5] and [18, Sect.2.2] with the definition of the weak subsolution and the derivation property of $\mu^{(\tau)}_{\langle u,v\rangle}$, one has

$$\begin{aligned} &\int_{\mathbb{R}^d} u(t_0,y)^p\psi(y)^2 m(\mathrm{d}y) + \frac{2(p-1)}{p}\int_{s_0}^{t_0}\int_{B(x,\delta r)} \psi(y)^2\chi(\tau)\mathrm{d}\mu^{(\tau)}_{\langle u^{p/2},u^{p/2}\rangle}(y)\mathrm{d}\tau \\ &\qquad \le \frac{2p}{p-1}\iint_{Q^-_\delta} u(\tau,y)^p\chi(\tau)\mathrm{d}\mu^{(\tau)}_{\langle\psi,\psi\rangle}(y)\mathrm{d}\tau + \iint_{Q^-_\delta} u(\tau,y)^p\psi(y)^2\chi'(\tau)m(\mathrm{d}y)\mathrm{d}\tau. \end{aligned}$$

Let

$$\lambda(Q_\delta^-) = \inf_{\tau\in I_\delta^-} \lambda(\tau, B(x,\delta r)) \quad \text{and} \quad \Lambda(Q_\delta^-) = \sup_{\tau\in I_\delta^-} \lambda(\tau, B(x,\delta r)).$$

Since t_0 is an arbitrary point of $I_{\delta'}^-$ and $\lambda(s,t,D)\mu^{(0)}_{\langle w,w\rangle}(\mathrm{d}y) \le \mu^{(\tau)}_{\langle w,w\rangle}(\mathrm{d}y) \le \Lambda(s,t,D)\mu^{(0)}_{\langle w,w\rangle}(\mathrm{d}y)$ for any $(\tau,y)\in Q_\delta^-$ and $w\in L^1_{\mathrm{loc}}([0,\infty);F)$, we have

$$\left(\sup_{\tau\in I_{\delta'}^-}\int_{\mathbb{R}^d} u(\tau,y)^p\psi(y)^2 m(\mathrm{d}y)\right)\vee\left(\lambda(s,t,D)\iint_{Q_{\delta'}^-}\mathrm{d}\mu^{(0)}_{\langle u^{p/2},u^{p/2}\rangle}(y)\mathrm{d}\tau\right)$$
$$\le \frac{K_1}{(\delta-\delta')^2r^2}\left(\Lambda(s,t,D)+\frac{1}{\eta}\right)\iint_{Q_\delta^-} u(\tau,y)^p m(\mathrm{d}y)\mathrm{d}\tau \tag{60}$$

for some constant $K_1 = K_1(p)$.

Put $\theta = 1+2/\nu$, $w = u^{p/2}\psi$ and $V(x,r) = m(B(x,r))$. By the Hölder inequality, for any $\nu \ge d$ with $\nu > 2$,

$$\int_{Q_\delta^-} w^{2\theta}\mathrm{d}m\mathrm{d}\tau \le \left(\sup_{\tau\in I_\delta^-}\int_{B(x,r)} w(\tau,\cdot)^2\mathrm{d}m\right)^{2/\nu}\int_{I_\delta^-}\left(\int_{B(x,\delta r)} w^{2\nu/(\nu-2)}\mathrm{d}m\right)^{(\nu-2)/\nu}\mathrm{d}\tau. \tag{61}$$

Further, by the Sobolev inequality,

$$\begin{aligned}\left(\int_{B(x,r)} w^{2\nu/(\nu-2)}\mathrm{d}m\right)^{(\nu-2)/\nu} &\le \frac{C_S r^2}{V(x,r)^{2/\nu}}\int_{B(x,r)}\mathrm{d}\mu^{(0)}_{\langle w,w\rangle}\\ &\le \frac{2C_S r^2}{V(x,r)^{2/\nu}}\left(\int_{B(x,r)}\mathrm{d}\mu^{(0)}_{\langle u^{p/2},u^{p/2}\rangle}+\frac{1}{(\delta-\delta')^2r^2}\int_{B(x,r)} u^p\mathrm{d}m\right)\end{aligned} \tag{62}$$

with the Sobolev constant C_S independent of $B(x,r)$ (cf. [14, Theorem 5.2.3]). Then, by combining the inequalities (60), (61) and (62) with the fact that $\psi = 1$ on $B(x,\delta' r)$, we obtain

$$\iint_{Q_{\delta'}^-} u^{p\theta}\mathrm{d}m\mathrm{d}\tau \le \frac{K_2 r^2}{V(x,r)^{2/\nu}}\frac{(\Lambda(s,t,D)+1/\eta)^\theta}{\lambda(s,t,D)}\frac{1}{(\delta-\delta')^{2\theta}r^{2\theta}}\left(\iint_{Q_{\delta'}^-} u^p\mathrm{d}m\mathrm{d}\tau\right)^\theta \tag{63}$$

for some constant K_2. Applying (63) to u^{θ^i}, δ_{i+1} and δ_i determined by

$$\delta_0 = \delta, \quad \delta_{i+1} = \delta_i - (\delta-\delta')2^{-(i+1)}$$

instead of u, δ' and δ, respectively, and making use of the Moser iteration, this yields the following result (cf. [18, Theorem 2.1], [10, Theorem 1.6]) :

Proposition A.1. *Assume that $p \ge 2$, $0 < \delta' < \delta \le 1$ and $Q_1^- \subset (s,t)\times D$. Then, for any non-negative locally bounded weak subsolution u of the heat equation for $E^{(\tau)}$ on $(s,t)\times D$, there exists a constant $K_3 = K_3(p,\nu)$ such that*

$$\sup_{Q_{\delta'}^-} u^p \le \frac{K_3}{r^2V(x,r)(\delta-\delta')^{\nu+2}}\frac{(\Lambda(s,t,D)+1/\eta)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}}\iint_{Q_\delta^-} u^p\mathrm{d}m\mathrm{d}\tau \tag{64}$$

Further, let $\widehat{u}$ is a non-negative locally bounded weak co-subsolution on $(s,t)\times D$. Then, similarly to Proposition A.1, it also holds that for $p \geq 2$ and $Q_1^+ \subset (s,t)\times D$, there exists a constant $K_4 = K_4(p,\nu)$ such that

$$\sup_{Q_{\delta'}^+} \widehat{u}^p \leq \frac{K_4}{r^2 V(x,r)(\delta-\delta')^{\nu+2}} \frac{(\Lambda(s,t,D)+1/\eta)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}} \iint_{Q_\delta^+} \widehat{u}^p \mathrm{d}m\mathrm{d}\tau. \tag{65}$$

Note that the inequalities (64) and (65) also hold for $0 < p < 2$ with $p \neq 1$ by changing the constants K_3 and K_4. Furthermore, if u (resp. $\widehat{u}$) is a non-negative locally bounded weak super-solution (resp. co-supersolution), then by taking $u_\varepsilon := u \vee \varepsilon$ (resp. $\widehat{u}_\varepsilon := \widehat{u} \vee \varepsilon$) for any $\varepsilon > 0$ instead of u (resp. $\widehat{u}$) and changing the constant K_3 (resp. K_4), the inequality (64) (resp. (65)) holds for $p < 0$ (cf. [18, Theorem 2.1], [10, Theorem 5.3]).

Set $\mathrm{d}\overline{\mu}(\tau,y) := \mathrm{d}m(y)\mathrm{d}\tau$ and $I = (a-\eta r^2, a+\eta r^2)$ for $\eta > 0$ and $r > 0$.

Lemma A.1. *([10, Lemma 5.1]) Let u be a locally bounded non-negative weak supersolution of the heat equation for $E^{(\tau)}$ on $(s,t)\times D$. For $x \in D$, take $r > 0$ such that $B(x,2r) \subset D$. Then, for any $\xi > 0$*

$$\overline{\mu}\left(\left\{(\tau,y)\in Q_\delta^+ : \log u_\varepsilon(\tau,y) < -\xi - c\right\}\right) \leq C_1 |I| V(x,r)\xi^{-1} \tag{66}$$

$$\overline{\mu}\left(\left\{(\tau,y)\in Q_\delta^- : \log u_\varepsilon(\tau,y) > \xi - c'\right\}\right) \leq C_1 |I| V(x,r)\xi^{-1}, \tag{67}$$

where the constants c and c' are given by

$$c = \frac{2\eta\Lambda(s,t,D)}{\delta^d(1-\delta)^2} + W_\varepsilon(a), \quad c' = -\frac{2\eta\Lambda(s,t,D)}{\delta^d(1-\delta)^2} + W_\varepsilon(a) \tag{68}$$

with $W_\varepsilon(\tau) = -\frac{1}{\int_{\mathbb{R}^d}\phi^2 \mathrm{d}m}\int_{\mathbb{R}^d}\log u_\varepsilon(\tau,\cdot)\phi^2 \mathrm{d}m$ for $\phi = (1-|x-\cdot|/r)\vee 0$, and the constant C_1 is given by

$$C_1 = \frac{C_P}{(1-\delta)^2\eta\lambda(s,t,D)} \tag{69}$$

with the Poincaré constant C_P satisfying (71) *below.*

Proof. For reader's convenience, we shall give shortly a proof of this lemma (see also [10, Lemma 6.2]). Let u be a weak supersolution of the heat equation for $E^{(\tau)}$ on $(s,t)\times D$. Then, for any $\tau \in (a, a+\eta r^2)$ and $\varepsilon > 0$

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^d}\log u_\varepsilon(\tau,\cdot)\phi^2\mathrm{d}m &= -\int_{\mathbb{R}^d}\frac{\partial u_\varepsilon(\tau,\cdot)}{\partial\tau}\frac{1}{u_\varepsilon(\tau,\cdot)}\phi^2\mathrm{d}m \\
&\leq \int_{\mathbb{R}^d}\mathrm{d}\mu^{(\tau)}_{\langle u_\varepsilon, \frac{\phi^2}{u_\varepsilon}\rangle} = -\int_{\mathbb{R}^d}\phi^2\mathrm{d}\mu^{(\tau)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle} + 2\int_{\mathbb{R}^d}\phi\,\mathrm{d}\mu^{(\tau)}_{\langle\log u_\varepsilon,\phi\rangle} \\
&\leq -\int_{\mathbb{R}^d}\phi^2\mathrm{d}\mu^{(\tau)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle} + \frac{1}{2}\int_{\mathbb{R}^d}\phi^2\mathrm{d}\mu^{(\tau)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle} + 2\int_{\mathbb{R}^d}\mathrm{d}\mu^{(\tau)}_{\langle\phi,\phi\rangle} \\
&= -\frac{1}{2}\int_{\mathbb{R}^d}\phi^2\mathrm{d}\mu^{(\tau)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle} + 2\int_{\mathbb{R}^d}\mathrm{d}\mu^{(\tau)}_{\langle\phi,\phi\rangle}.
\end{aligned}$$

Since $\mathrm{d}\mu^{(0)}_{\langle\phi,\phi\rangle} \leq (1/r^2)1_{B(x,r)}\mathrm{d}m$, the above inequality yields that

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^d}\log u_\varepsilon(\tau,\cdot)\phi^2\mathrm{d}m + \frac{\lambda(s,t,D)}{2}\int_{\mathbb{R}^d}\phi^2\mathrm{d}\mu^{(0)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle} \leq \frac{2\Lambda(s,t,D)}{r^2}V(x,r). \tag{70}$$

Note that under our present setting, the following weighted Poincaré inequality holds ([14, Theorem 5.3.4]): there is a constant $C_P > 0$ independent of x and r such that

$$\int_{B(x,r)}\left|-\log u_\varepsilon(\tau,\cdot) - W_\varepsilon(\tau)\right|^2\phi^2\mathrm{d}m \leq C_P r^2\int_{B(x,r)}\phi^2\mathrm{d}\mu^{(0)}_{\langle\log u_\varepsilon,\log u_\varepsilon\rangle}. \tag{71}$$

Using this, (70) can be written as

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{B(x,r)}\log u_\varepsilon(\tau,\cdot)\phi^2\mathrm{d}m &+ \frac{\lambda(s,t,D)}{2C_P r^2}\int_{B(x,r)}\left|-\log u_\varepsilon(\tau,\cdot) - W_\varepsilon(\tau)\right|^2\phi^2\mathrm{d}m \\ &\leq \frac{2\Lambda(s,t,D)}{r^2}V(x,r).\end{aligned} \tag{72}$$

Set

$$\begin{aligned} w(\tau,z) &= -\log u_\varepsilon(\tau,z) - \frac{2V(x,r)}{r^2\int_{\mathbb{R}^d}\phi^2\mathrm{d}m}\Lambda(s,t,D)(\tau-a) \\ \overline{W}_\varepsilon(\tau) &= W_\varepsilon(\tau) - \frac{2V(x,r)}{r^2\int_{\mathbb{R}^d}\phi^2\mathrm{d}m}\Lambda(s,t,D)(\tau-a). \end{aligned}$$

Then, we see by (72) that

$$\frac{\mathrm{d}\overline{W}_\varepsilon(\tau)}{\mathrm{d}\tau} + \frac{\lambda(s,t,D)}{2C_P r^2\int_{\mathbb{R}^d}\phi^2\mathrm{d}m}\int_{B(x,r)}\left|w(\tau,\cdot) - \overline{W}_\varepsilon(\tau)\right|^2\phi^2\mathrm{d}m \leq 0.$$

In particular, $\overline{W}_\varepsilon(\tau)$ is non-increasing relative to τ. For fixed $\delta \in (0,1)$ and for any $\xi > 0$, put

$$\Omega^+_\tau(\delta,\xi) := \{z \in B(x,\delta r) : w(\tau,z) > \xi + \overline{W}_\varepsilon(a)\}.$$

Since $w(\tau,z) - \overline{W}_\varepsilon(\tau) > \xi + \overline{W}_\varepsilon(a) - \overline{W}_\varepsilon(\tau) \geq 0$ and $\phi(z) \geq 1-\delta$ for $z \in \Omega^+_\tau(\delta,\xi)$ and $\tau \geq a$, it follows that

$$\frac{\mathrm{d}\overline{W}_\varepsilon(\tau)}{\mathrm{d}\tau} + \frac{(1-\delta)^2\lambda(s,t,D)}{2C_P r^2 V(x,r)}\left|\xi + \overline{W}_\varepsilon(a) - \overline{W}_\varepsilon(t)\right|^2 m\left(\Omega^+_\tau(\delta,\xi)\right) \leq 0,$$

or equivalently,

$$\frac{(1-\delta)^2\lambda(s,t,D)}{2C_P r^2 V(x,r)}\left|\xi + \overline{W}_\varepsilon(a) - \overline{W}_\varepsilon(t)\right|^2 m\left(\Omega^+_\tau(\delta,\xi)\right) \leq \frac{\mathrm{d}}{\mathrm{d}\tau}\left|\xi + \overline{W}_\varepsilon(a) - \overline{W}_\varepsilon(\tau)\right|.$$

This implies

$$m\left(\Omega_\tau^+(\delta,\xi)\right) \le \frac{2C_P r^2 V(x,r)}{(1-\delta)^2\lambda(s,t,D)} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{1}{\left|\xi + \overline{W}_\varepsilon(a) - \overline{W}_\varepsilon(\tau)\right|}\right).$$

Then, by integrating both sides above on $(a, a+\delta\eta r^2)$ relative to τ, we obtain

$$\overline{\mu}\left(\left\{(\tau,z) \in Q_\delta^+ : w(\tau,z) > \xi + \overline{W}_\varepsilon(a)\right\}\right) \le \frac{2C_P r^2 V(x,r)}{(1-\delta)^2\lambda(s,t,D)}\xi^{-1}. \tag{73}$$

Further, since $|I| = 2\eta r^2$ and

$$\int_{\mathbb{R}^d} \phi^2 \mathrm{d}m \ge \int_{B(x,\delta r)} \phi^2 \mathrm{d}m \ge (1-\delta)^2 V(x,\delta r) \ge (1-\delta)^2\delta^d V(x,r),$$

we see that for $\tau \in I_\delta^+$

$$\frac{2V(x,r)}{r^2\int_{\mathbb{R}^d}\phi^2\mathrm{d}m}\Lambda(s,t,D)(\tau - a) \le \frac{2\Lambda(s,t,D)\eta}{(1-\delta)^2\delta^d}.$$

From this with (73), we can confirm the assertion (66). The proof of the assertion (67) is similar. We omit the details. □

Now, we shall prove the parabolic Harnack inequality (cf. [1, 10, 11, 19]). First, we present the following general result given by [10, Lemma 5.2]. Let $U_\delta \subset D$ and $J_\delta \subset [0,\infty)$ be families of sets such that $U_{\delta'} \subset U_\delta$ and $J_{\delta'} \subset J_\delta$ for $0 < \delta' < \delta \le 1$.

Lemma A.2. *Fix $\delta^* \in (0,1)$. Assume that a positive measurable function f on $J_1 \times U_1$ satisfies*

$$\sup_{J_{\delta'}\times U_{\delta'}} f \le \left(\frac{C_2}{(\delta-\delta')^\gamma}\frac{1}{|J_1|m(U_1)}\iint_{J_\delta\times U_\delta} f^p \mathrm{d}\tau\mathrm{d}m\right)^{1/p}$$

for some $\gamma > 0$, $C_2 > 1$ and for all $0 < \delta^* \le \delta' < \delta \le 1$ and $p \in (0, 1-\varepsilon)$ for some $\varepsilon \in (0,1)$. Further, assume that there exists a constant $C_3 > 1$ such that

$$\overline{\mu}\left(\{\log f > \xi\}\right) \le C_3\frac{|J_1|m(U_1)}{\xi} \tag{74}$$

for all $\xi > 0$. Then there exists a constant $K_5 > 0$ such that

$$\sup_{J_{\delta^*}\times U_{\delta^*}} f \le K_5 C_2^2 C_3. \tag{75}$$

For fixed $0 < \delta_1 < \delta_2 < 1$, let

$$Q^- := (a - \delta_2 \eta r^2, a - \delta_1 \eta r^2) \times B(x, r),$$
$$Q^+ := (a + \delta_1 \eta r^2, a + \delta_2 \eta r^2) \times B(x, r).$$

Let $v_\varepsilon = u_\varepsilon e^{c'}$ with c' given in (68). By virtue of Proposition A.1 and the comments after that, v_ε satisfies the condition of Lemma A.2 for $J_\delta \times U_\delta = (a - (\delta_2 - \delta_1 + \delta)\eta r^2, a - \delta_1 \eta r^2) \times B(x, \delta r)$ with the constant $\gamma = \nu + 2$ and

$$C_2 = \frac{K_3 \eta (\Lambda(s,t,D) + 1/\eta)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}}.$$

Further, (74) is satisfied by (67) with $C_3 = C_1$ given in (69). Therefore, by putting $\delta^* = \delta_1$ in (75), we obtain

$$\sup_{(\tau,z)\in Q^-} u_\varepsilon(\tau, z) e^{c'} \le K_6 C_1 C_2^2 \tag{76}$$

for some constant $K_6 > 0$. Similarly, applying Lemma A.2 to $f = (u_\varepsilon e^c)^{-1}$ with c given in (68), it follows that

$$\sup_{(\tau,z)\in Q^+} (u_\varepsilon(\tau, z) e^c)^{-1} \le K_7 C_1 C_2^2 \tag{77}$$

for some constant $K_7 > 0$. Now, by combining (76) and (77), we can obtain the following parabolic Harnack inequality:

Theorem A.1. *There exist positive constants K_8 and K_9 such that for any non-negative locally bounded weak solution u of the heat equation for $E^{(\tau)}$ on $(s,t) \times D$,*

$$\sup_{(\tau,z)\in Q^-} u(\tau, z) \le C_\eta(s,t) \inf_{(\tau,z)\in Q^+} u(\tau, z), \tag{78}$$

where

$$C_\eta(s,t) = K_8 e^{K_9 \eta \Lambda(s,t,D)} \frac{(\Lambda(s,t,D) + 1/\eta)^{2\nu+4} \eta^2}{\lambda(s,t,D)^{2\nu+2}}.$$

We are ready to prove Proposition 3.1.

Proof of Proposition 3.1. For notational convenience, we take as $R = R(t) = 2\varphi(t)$, $D = D(t) = B(5\varphi(t))$, $\eta = 1/(2\Lambda(s,t,D))$ and $x, y \in B(R(t))$. For any non-negative function $f \in L^2(D)$ and $s \le \sigma < \tau \le t$, let

$$v(\tau, w) := v_\beta(\sigma; \tau, w) = \int_D \overline{p}(\sigma, z; \tau, w) e^{-\beta|x-z|} f(z) m(\mathrm{d}z), \quad \beta \in \mathbb{R}$$

with the transition density $\overline{p}(\sigma, z; \tau, w)$ of the diffusion process $\overline{\mathbb{M}}$ on $\mathbb{R}^d$ given in Section 4. Then, by [18, Lemma 1.7],

$$\left\|e^{\beta|x-\cdot|}v(\tau,\cdot)\right\|^2_{L^2(D)} \leq e^{2\beta^2(\tau-\sigma)\Lambda(s,t,D)}\|f\|^2_{L^2(D)}. \tag{79}$$

Let $\widehat{Q}_1^- = (t-\eta r^2, t)\times B(y,r)$ for $r \leq R(t)/2$. Since $|z-x| \geq |x-y| - r$ for $z \in B(y,r)$, we see from (64) and (79) that

$$\begin{aligned} v^2(t,y) &\leq \frac{K_3'}{r^2V(y,r)(1-\delta)^{\nu+2}}\frac{\Lambda(s,t,D)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}}\iint_{\widehat{Q}_1^-} v^2 \mathrm{d}m\mathrm{d}\tau \\ &\leq \frac{K_3'}{V(y,r)(1-\delta)^{\nu+2}}\frac{\Lambda(s,t,D)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}}e^{-2\beta(|x-y|-r)}e^{2\beta^2(t-\sigma)\Lambda(s,t,D)}\eta\|f\|^2_{L^2(D)} \end{aligned} \tag{80}$$

for a constant $K_3' > 0$. On the other hand, suppose that f is supported by $B(x,r)$. Then

$$v^2(t,y) \geq e^{-2\beta r}\left(\int_{B(x,r)} \overline{p}(\sigma,z;t,y)f(z)m(\mathrm{d}z)\right)^2. \tag{81}$$

Combining (80) and (81), and taking the supremum relative to $f \in L^2(B(x,r))$ with $\|f\|_{L^2(B(x,r))} = 1$, it follows that

$$\int_{B(x,r)} \overline{p}(\sigma,z;t,y)^2 m(\mathrm{d}z) \leq \frac{K_3''}{V(y,r)(1-\delta)^{\nu+2}}\frac{\Lambda(s,t,D)^{\nu/2}}{\lambda(s,t,D)^{\nu/2}}e^{-2\beta|x-y|+2\beta r}e^{2\beta^2(t-\sigma)\Lambda(s,t,D)}$$

for a constant $K_3'' > 0$. Further, since $\widehat{u}(\sigma,z) = \overline{p}(\sigma,z;t,y)$ is a co-soulution for $(\sigma,z) \in \widehat{Q}_1^+ = (s, s+\eta r^2)\times B(x,r)$, we see by the dual result (65) of Proposition A.1 that there is a constant $K_{10} > 0$ such that

$$\begin{aligned} \overline{p}(s,x;t,y)^2 &\leq \frac{K_4'}{r^2V(x,r)(1-\delta)^{\nu+2}}\frac{\Lambda(s,t,D)^{1+\nu/2}}{\lambda(s,t,D)^{\nu/2}}\iint_{\widehat{Q}_1^+} \overline{p}(\sigma,z;t,y)^2 m(\mathrm{d}z)\mathrm{d}\sigma \\ &\leq \frac{K_{10}}{V(x,r)V(y,r)(1-\delta)^{2(\nu+2)}}\left(\frac{\Lambda(s,t,D)}{\lambda(s,t,D)}\right)^{\nu}e^{-2\beta|x-y|+2\beta r}e^{2\beta^2(t-s)\Lambda(s,t,D)}. \end{aligned}$$

Finally, by putting

$$\beta = \frac{|x-y|}{2(t-s)\Lambda(s,t,D)} \quad \text{and} \quad r = \sqrt{(t-s)\Lambda(s,t,D)} \wedge \frac{(t-s)\Lambda(s,t,D)}{2|x-y|},$$

we obtain (25). The last assertion of Proposition 3.1 is clear because $p^D(s,x;t,y) \leq \overline{p}(s,x;t,y)$. □

In the sequel, we consider that $D = D(t) = B(5R(t)/2)$ and $\eta = \eta(t) = 1/(2\Lambda(s,t,D(t)))$ with $R(t) = 2\varphi(t)$. For the function $\psi(t)$ given by (26) satisfying Assumption (A), put

$$\eta_\psi = \frac{1}{2\psi(t)\Lambda(s,t,D(t))} \quad \text{and} \quad r_\psi(s,t) = \sqrt{2(t-s)\Lambda(s,t,D(t))\psi(t)}.$$

Clearly, $r_\psi(s,t) = r(s,t) = \sqrt{2(t-s)\Lambda(s,t,D(t))}$ when $\psi = 1$. Further, let $C_\eta(s,t)$ and $C_{\eta_\psi}(s,t)$ be the parabolic Harnack constants corresponding to η and η_ψ, respectively, that is,

$$C_{\eta_\psi}(s,t) = C_4\left(\frac{\Lambda(s,t,D(t))\psi(t)}{\lambda(s,t,D(t))}\right)^{2\nu+2}$$

and $C_\eta(s,t)$ is defined similarly by putting $\psi = 1$. Then $C_\eta(s,t) = C_4 C_H(s,t)$.

Lemma A.3. *Suppose that Assumption (A) holds. For $\gamma > 1$, let $\{s_j\}$ be a sequence of real numbers defined by $s_j = \gamma^j s$. Then, there exists $T > 0$ such that for any $s \geq T$,*

$$\int_{B(y,r_\psi(s_1,s_2))} p^{D(s_3)}(s_1,z;s_2,y)m(\mathrm{d}z) \geq \frac{1}{2}$$

for all $y \in B(R(s_3))$.

Proof. In the present proof, we simply write $D(s_3)$ and $r_\psi(s_2,s_3)$ as D and r_ψ, respectively. Put $\overline{p}(s,A;\tau,y) := \int_A \overline{p}(s,z;\tau,y)m(\mathrm{d}z)$ for $A \subset \mathbb{R}^d$. For any $y \in B(R(s_3))$ and $j \geq 1$, let

$$A(j,r_\psi) := B(y,(j+1)r_\psi) \setminus B(y,jr_\psi)$$

and $u_j(\tau,z) := \overline{p}(s_1,A(j,r_\psi);\tau,z)$. Then, the parabolic Harnack inequality implies

$$u_j(s_2,y) \leq \frac{C_\eta(s_2,s_3)}{V(y,r(s_2,s_3))}\int_{B(y,r(s_2,s_3))} u_j(s_3,z)m(\mathrm{d}z).$$

On the other hand, by virtue of [18, Theorem 0.1],

$$(\overline{p}(s,A;\tau,\cdot),1_B) \leq \sqrt{m(A)}\sqrt{m(B)}\exp\left(-\frac{d(A,B)^2}{4(\tau-s)\Lambda(s,\tau,D)}\right), \tag{82}$$

where $d(A,B) = \inf\{|x-y| : x \in A, y \in B\}$ for $A, B \subset \mathbb{R}^d$. Then, by applying (82) to $A = A(j,r_\psi)$ and $B = B(y,r(s_2,s_3))$,

$$\begin{aligned}
&\frac{C_\eta(s_2,s_3)}{V(y,r(s_2,s_3))}\int_{B(y,r(s_2,s_3))}\overline{p}(s_1,A(j,r_\psi);s_3,z)m(\mathrm{d}z)\\
&\qquad\leq C_\eta(s_2,s_3)\frac{V(A(j,r_\psi))^{1/2}}{V(y,r(s_2,s_3))^{1/2}}\exp\left(-\frac{d(A(j,r_\psi),B(y,r(s_2,s_3)))^2}{2r(s_1,s_3)^2}\right).
\end{aligned}$$

Let $\varepsilon_0 \in (0,1)$ be given in (A2) of Assumption (A). Take $\varepsilon > 0$ such that $(1-2\varepsilon)^3 > 1-\varepsilon_0$. Then $r(s_2,s_3) \leq \varepsilon r_\psi$ for large enough s and hence $d(A(j,r_\psi),B(y,r(s_2,s_3))) \geq (j-\varepsilon)r_\psi$. Further, since $r(s_2,s_3)/r(s_1,s_3) \geq 1/\sqrt{2}$,

$$\begin{aligned}
u_j(s_2,y) &\leq C_\eta(s_2,s_3)\frac{V(A(j,r_\psi))^{1/2}}{V(y,r(s_2,s_3))^{1/2}}\exp\left(-\frac{(j-\varepsilon)^2 r_\psi^2}{2r(s_1,s_3)^2}\right)\\
&\leq K_{11}C_\eta(s_2,s_3)\frac{(j+1)^{d/2}r_\psi^{d/2}}{r(s_2,s_3)^{d/2}}\exp\left(-\frac{(j-\varepsilon)^2 r(s_2,s_3)^2\psi(s_3)}{2r(s_1,s_3)^2}\right)\\
&\leq K_{11}C_\eta(s_2,s_3)(j+1)^{d/2}\psi(s_3)^{d/4}\exp\left(-\frac{(1-\varepsilon)(j-\varepsilon)^2\psi(s_3)}{4}\right).
\end{aligned}$$

Therefore, by adding the both sides above relative to $j \geq 1$ and noting $\psi(s_3) \leq e^{\varepsilon(j-\varepsilon)^2\psi(s_3)/4}$ for all $j \geq 1$ for large s_3, we have

$$\begin{aligned}\int_{D\setminus B(y,r_\psi)} \overline{p}(s_1,z;s_2,y)m(\mathrm{d}z) &= \sum_{j=1}^{\infty}\int_D \overline{p}(s_1,z;s_2,y)1_{A(j,r_\psi)}(z)m(\mathrm{d}z) = \sum_{j=1}^{\infty} u_j(s_2,y)\\ &\leq K_{11}C_\eta(s_2,s_3)\psi(s_3)^{d/4}\sum_{j=1}^{\infty}(j+1)^{d/2}\exp\left(-\frac{(1-\varepsilon)(j-\varepsilon)^2\psi(s_3)}{4}\right)\\ &\leq K_{12}C_\eta(s_2,s_3)\int_1^\infty (z+1)^{d/2}\exp\left(-\frac{(1-2\varepsilon)(z-\varepsilon)^2\psi(s_3)}{4}\right)\mathrm{d}z\\ &\leq K_{13}C_\eta(s_2,s_3)\exp\left(-\frac{(1-2\varepsilon)^3\psi(s_3)}{4}\right)\\ &\leq K_{14}\exp\left(-\frac{(1-\varepsilon_0)}{4}\psi(s_3)+2(\nu+1)\log\Lambda(s_2,s_3,D)\right)\\ &\equiv H(s_3,r_\psi),\end{aligned}$$

where we used (A2) in Assumption (A) in the last inequality above. This can be made smaller than 1/4 for all $s \geq T_1$ by taking large T_1. Thus we have for such s_3 that

$$\int_{B(y,r_\psi)} \overline{p}(s_1,z;s_2,y)m(\mathrm{d}z) \geq 1 - H(s_3,r_\psi) \geq \frac{3}{4}$$

in view of the conservativeness of $\overline{\mathbb{M}}$. By [15, Lemma 3],

$$\overline{\mathbb{P}}_{(s_1,x)}(\tau_D \leq s_2) \leq \overline{\mathbb{P}}_{(s_1,x)}(\tau_{B(x,r_\psi)} \leq s_2) \leq \frac{H(s_3,r_\psi/2)}{1-H(s_3,r_\psi/2)}. \tag{83}$$

Further, by the strong Markov property,

$$p^D(s_1,x;s_2,y) = \overline{p}(s_1,x;s_2,y) - \overline{\mathbb{E}}_{(s_1,x)}\left[\overline{p}(\tau_D,X_{\tau_D};s_2,y) : \tau_D < s_2\right]. \tag{84}$$

For $y \in B(R(s_3))$ and $\tau_D \in (s_1,s_2)$, $|X_{\tau_D} - y| \geq R(s_3)$ and $(s_2-\tau_D)\Lambda(\tau_D,s_2,D) \leq (s_2 - s_1)\Lambda(s_1,s_2,D)$. Further, note that

$$\frac{2|X_{\tau_D}-y|}{\sqrt{(s_2-\tau_D)\Lambda(\tau_D,s_2,D)}} \geq \frac{2R(s_3)}{R(s_3)} > 1 \quad \text{and} \quad \frac{R(s_3)^2}{4(s_2-s_1)\Lambda(s_1,s_2,D)} \geq 2\psi(s_3).$$

By (25)) with the fact that $\xi^{2d}e^{-\xi^2}$ is decreasing relative to large ξ, we then have

$$\begin{aligned}
\overline{p}(\tau_D, X_{\tau_D}; s_2, y) &\leq \frac{K_1}{((s_2-\tau_D)\Lambda(\tau_D, s_2, D))^{d/2}} \left(\frac{\Lambda(\tau_D, s_2, D)}{\lambda(\tau_D, s_2, D)}\right)^{\nu/2} \\
&\quad \times \left(\frac{4|X_{\tau_D}-y|^2}{(s_2-\tau_D)\Lambda(\tau_D, s_2, D)}\right)^{d/2} \exp\left(-\frac{|X_{\tau_D}-y|^2}{4(s_2-\tau_D)\Lambda(\tau_D, s_2, D)}\right) \\
&\leq \frac{K_{15}\Lambda(s_1, s_2, D)^{\nu/2}}{R(s_3)^d} \left(\frac{|X_{\tau_D}-y|^2}{(s_2-\tau_D)\Lambda(\tau_D, s_2, D)}\right)^{d} \exp\left(-\frac{|X_{\tau_D}-y|^2}{4(s_2-\tau_D)\Lambda(\tau_D, s_2, D)}\right) \\
&\leq \frac{K_{15}\Lambda(s_1, s_2, D)^{\nu/2}}{R(s_3)^d} \left(\frac{R(s_3)^2}{(s_2-s_1)\Lambda(s_1, s_2, D)}\right)^{d} \exp\left(-\frac{R(s_3)^2}{4(s_2-s_1)\Lambda(s_1, s_2, D)}\right) \\
&\leq \frac{K_{16}\Lambda(s_1, s_2, D)^{\nu/2}}{R(s_3)^d} \psi(s_3)^d e^{-2\psi(s_3)}.
\end{aligned}$$

From this together with (83), (A2) in Assumption (A) and the fact that $(r_\psi/R(s_3))^d \leq (1/2)^d$, we obtain

$$\begin{aligned}
&\int_{B(y,r_\psi)} \overline{\mathbb{E}}_{(s_1,z)}\left[\overline{p}(\tau_D, X_{\tau_D}; s_2, y) : \tau_D < s_2\right] m(\mathrm{d}z) \\
&\leq \frac{K_{17}(r_\psi)^d\Lambda(s_1, s_2, D)^{\nu/2}}{R(s_3)^d} \psi(s_3)^d e^{-2\psi(s_3)} \frac{H(s_3, r_\psi/2)}{1-H(s_3, r_\psi/2)} \\
&\leq K_{18}\psi(s_3)^d \exp\left(-2\psi(s_3) + \frac{\nu}{2}\log\Lambda(s_1, s_2, D)\right) \frac{H(s_3, r_\psi/2)}{1-H(s_3, r_\psi/2)} \\
&\leq K_{18}\psi(s_3)^d \exp\left(-2\psi(s_3) + \frac{\nu}{2(1-\varepsilon_0)}\log\Lambda(0, s_3, D)\right) \frac{H(s_3, r_\psi/2)}{1-H(s_3, r_\psi/2)} \\
&\leq 1/4
\end{aligned}$$

for all $s \geq T_2$ with large T_2. Hence, the assertion of the lemma follows from (84) by taking $T = T_1 \vee T_2$. $\square$

Finally, we give a proof of Proposition 3.2

Proof of Proposition 3.2. Let $T > 0$, $\gamma > 1$ and $\psi(t)$ be the function given by (26) satisfying Assumption (A). For $T \leq s < t \leq \gamma s$ and $x, y \in B(R(t))$, let k be the positive integer satisfying $\gamma(k-1)r(s,t)^2 \leq |x-y|^2 < \gamma(k)r(s,t)^2$, where $\gamma(j) = [\gamma^j]$ is the integer part of γ^j $(j \geq 1)$. Put

$$\begin{aligned}
s_0 = s, \quad s_j = s + j(t-s)/\gamma(k) \\
x_0 = x, \quad x_j = x + j(y-x)/\gamma(k).
\end{aligned}$$

Clearly, $s_{\gamma(k)} = t$, $x_{\gamma(k)} = y$ and

$$|x_j - x_{j-1}|^2 = \frac{|x-y|^2}{\gamma(k)^2} < \frac{r(s,t)^2}{\gamma(k)} = 2(s_j - s_{j-1})\Lambda(s, t, D(t)) := \overline{r}(s_{j-1}, s_j)^2.$$

Let $u(\tau,z) = p^D(s,x;\tau,z)$. By the parabolic Harnack inequality for $u(\tau,z)$ on $(s_{j-1},s_j) \times B(x_j,\overline{r}(s_{j-1},s_j))$ and $\eta = 1/2\Lambda(s,t,D(t))$,

$$u(t,y) \geq \frac{1}{C_\eta(s,t)} u(s_{\gamma(k)-1}, x_{\gamma(k)-1}) \geq \cdots \geq \frac{1}{C_\eta(s,t)^{\gamma(k)-2}} u(s_2,x_2). \tag{85}$$

Further, since $|x-x_2| < \sqrt{4(s_2-s)\Lambda(s,t,D(t))\psi(s_2)} := \overline{r}_\psi(s,s_2)$, applying parabolic Harnack inequality to $(\sigma,z) \mapsto p^D(\sigma,z;s_2,x_2)$,

$$p^D(s,x;s_2,x_2) \geq \frac{1}{C_{\eta_\psi}(s,t)V(x_2,\overline{r}_\psi(s,s_2))} \int_{B(x_2,\overline{r}_\psi(s,s_2))} p^D(s_1,z;s_2,x_2)m(\mathrm{d}z).$$

By Lemma A.3, the integral of the righthand side above is larger than $1/2$. Note that $\gamma(k)-1 \leq \gamma\gamma(k-1)+\gamma-1$. Hence, if s is large, then by (85) together with the relations $\gamma(k-1) \leq |x-y|^2/r(s,t)^2 < \gamma(k)$ and $\overline{r}_\psi(s,s_2) = r(s,t)\sqrt{2\psi(s_2)/\gamma(k)}$, we obtain

$$\begin{aligned} p^D(s,x;t,y) &\geq \frac{K_{19}}{C_\eta(s,t)^{\gamma(k)-2} C_{\eta_\psi}(s,t) V(x_2, r(s,t)\sqrt{2\psi(s_2)/\gamma(k)})} \\ &\geq \frac{K_{20}\gamma(k)^{d/2}}{\psi(t)^{2(\nu+1)+d/2} C_\eta(s,t)^{\gamma(k)-1} V(x_2,r(s,t))} \\ &\geq \frac{K_{20}\gamma(k)^{d/2}}{\psi(t)^{2(\nu+1)+d/2} C_\eta(s,t)^{\gamma\gamma(k-1)+\gamma-1} V(x_2,r(s,t))} \\ &\geq \frac{K_{20}|x-y|^d}{\psi(t)^{2(\nu+1)+d/2} r(s,t)^{2d} C_\eta(s,t)^{\gamma-1}} \exp\left(-\frac{\gamma|x-y|^2}{r(s,t)^2}\log C_\eta(s,t)\right). \end{aligned}$$

The proof is complete. □

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